Stochastic approximation algorithms with set-valued mean fields: *Theory and applications*

Arunselvan. R

Thesis Advisor: Prof. Shalabh Bhatnagar

Department of Computer Science and Automation, Indian Institute of Science.

arunselvan@csa.iisc.ernet.in

Quick introduction to stochastic approximation algorithms

Consider the following recursion in $\mathbb{R}^d$ $(d \geq 1)$:

$$x_{n+1} = x_n + a(n) \left[h(x_n) + M_{n+1}\right], \; for \; n \geq 0, \text{ where} \tag{1}$$

(i) $h : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz continuous function.

(ii) $a(n) > 0$, for all n, is the step-size sequence satisfying $\sum_{n=0}^{\infty} a(n) = \infty$ and $\sum_{n=0}^{\infty} a(n)^2 < \infty$.

(iii) M_n, $n \geq 1$, is a sequence of martingale difference terms that constitute the noise.

In 1996, **Benaïm** [1] showed that the asymptotic behavior of a stochastic recursive equation can be studied by analyzing the asymptotic behavior of the *associated o.d.e.*

Borkar-Meyn theorem for stochastic recursive inclusions, [3]

The objective is to develop sufficient conditions that are easily verifiable for both stability and convergence of set-valued dynamical systems given by:

$$x_{n+1} = x_n + a(n) \left[y_n + M_{n+1}\right], \; for \; n \geq 0, \text{ where} \tag{2}$$

$y_n \in h(x_n)$ and $h : \mathbb{R}^d \to \{subsets \; of \; \mathbb{R}^d\}$ is a Marchaud map.

Although there are two different set of assumptions in [3], we consider only one here.

Assumptions

h is a Marchaud map. The step-size and Martingale noise sequence satisfy the standard assumptions. Below we state the key assumptions of our paper, see [3].

For $c \geq 1$ and $x \in \mathbb{R}^d$, define $h_c(x) = \{y \mid cy \in h(cx)\}$. Further, for each $x \in \mathbb{R}^d$, define $h_\infty(x) := \overline{Liminf_{c\to\infty} h_c(x)}$ *i.e.* the closure of the *lower-limit* of $\{h_c(x)\}_{c\geq 1}$.

(A4) $h_\infty(x)$ is non-empty for all $x \in \mathbb{R}^d$. Further, the differential inclusion $\dot{x}(t) \in h_\infty(x(t))$ has an attracting set, $\mathcal{A}$, with $\overline{B}_1(0)$ as a subset of its fundamental neighborhood. This attracting set is such that $\mathcal{A} \subseteq B_1(0)$.

(A5) Let $c_n \geq 1$ be an increasing sequence of integers such that $c_n \uparrow \infty$ as $n \to \infty$. Further, let $x_n \to x$ and $y_n \to y$ as $n \to \infty$, such that $y_n \in h_{c_n}(x_n)$, $\forall n$, then $y \in h_\infty(x)$.

Define $\delta_1 := \sup_{x\in\mathcal{A}} \|x\|$ and pick real numbers δ_2, δ_3 and δ_4 such that $\sup_{x\in\mathcal{A}} \|x\| = \delta_1 < \delta_2 < \delta_3 < \delta_4 < 1$.

Outline of the proof

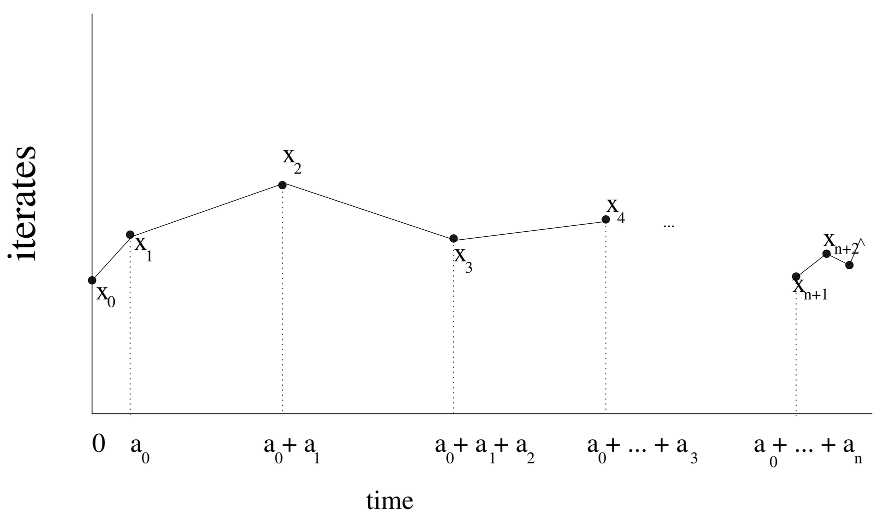

Figure 1: Linearly interpolated trajectory

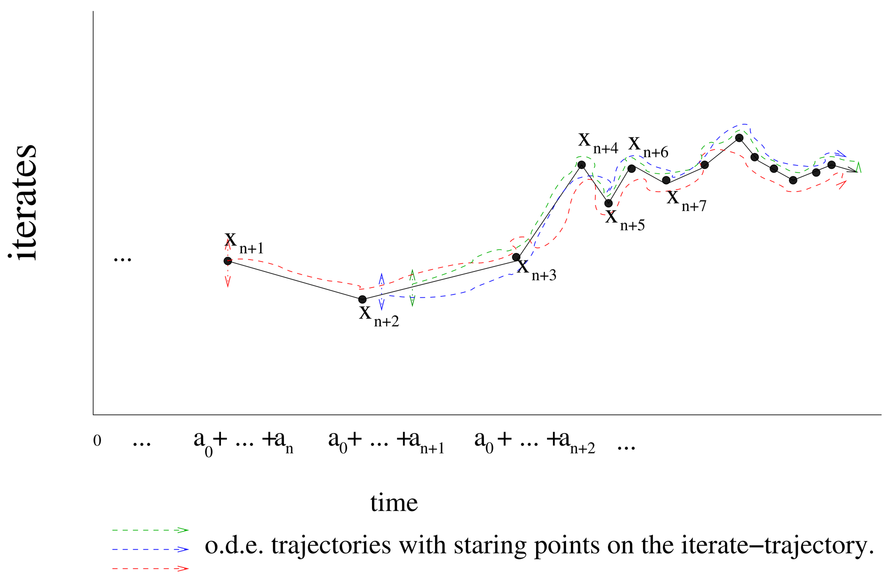

Figure 2: Tracking the associated o.d.e.

We provide a brief outline of our approach to prove the stability of a SRI under assumptions $(A1) - (A5)$.

- ✓ Divide the time line, $[0, \infty)$, approximately into intervals of length T.
- ✓ T is such that any solution to $\dot{x}(t) \in h_\infty(x(t))$ with starting point in the unit ball will be "inside" the unit ball and "close" to the attractor after time T.
- ✓ Construct the *linearly interpolated trajectory* from the given stochastic recursive inclusion. A sequence of 'rescaled' trajectories of length T is constructed as follows: At the beginning of each T-length interval we observe the trajectory to see if it is outside the unit ball, if so we scale it back to the boundary of the unit ball. This scaling factor is then used to scale the 'rest of the T-length trajectory'.
- ✓ To show that the iterates are bounded almost surely we need to show that the linearly interpolated trajectory does not 'run off' to infinity. To do so we assume that this is not true and show that there exists a subsequence of the rescaled T-length trajectories that has a solution to $\dot{x}(t) \in h_\infty(x(t))$ as a limit point in $C([0, T], \mathbb{R}^d)$.
- ✓ We choose and fix T such that any solution to $\dot{x}(t) \in h_\infty(x(t))$ with an initial value inside the unit ball is close to the origin at the end of time T. In this paper we choose $T = T(\delta_2 - \delta_1) + 1$.
- ✓ We then argue that the linearly interpolated trajectory is forced to make arbitrarily large 'jumps' within time T. The *Gronwall inequality* is then used to show that this is not possible.
- ✓ Once we prove stability of the recursion we invoke *Theorem 3.6 & Lemma 3.8* from **Benaïm, Hofbauer and Sorin** [2] to conclude that the limit set is a closed, connected, internally chain transitive and invariant set associated with $\dot{x}(t) \in h_\infty(x(t))$.

Application: The problem of approximate drift

- ✓ In practice the drift function cannot be calculated accurately. A natural question is the following: Are the iterates stable? If so, where do they converge.
- ✓ We use our framework to show that the algorithm with approximate drift is stable provided the algorithm with "accurate drift" was stable. Further, we show that the algorithm converges to a neighborhood of the "intended" set, where the neighborhood is dependent on the drift errors.

Gradient based learning algorithms with constant-error gradient estimators, [6]

- ✓ Implementations of stochastic gradient search algorithms such as back propagation typically rely on finite difference (FD) approximation methods. These methods are used to approximate the objective function gradient in **steepest descent** algorithms as well as the gradient and Hessian inverse in **Newton** based schemes.
- ✓ Hitherto in literature, the convergence analyses critically require that perturbation parameters in the estimators of the gradient/Hessian approach zero. However, in practice, the perturbation parameter is often held fixed to a 'small' constant resulting in constant-error estimates. item In [6], we present a framework to analyze the aforementioned.
- ✓ Easily verifiable conditions are presented for stability and convergence when using such FD estimators for the gradient/Hessian. In addition, our framework dispenses with a critical restriction on the step-sizes (learning rate) when using FD estimators, see [6] for details.

Stochastic recursive inclusion in two timescales with an application to the Lagrangian dual problem, [5]

We consider the following coupled iteration.

$$\begin{aligned} x_{n+1} &= x_n + a(n) \left[u_n + M^1_{n+1}\right], \\ y_{n+1} &= y_n + b(n) \left[v_n + M^2_{n+1}\right], \end{aligned} \tag{3}$$

where $u_n \in h(x_n, y_n)$, $v_n \in g(x_n, y_n)$ such that h and g are Marchaud maps. The step-size satisfies the standard assumptions and $\frac{b(n)}{a(n)} \to 0$. The iterates are assumed to be stable. The following is a key assumption that couples the x and the y iterates, see [5] for more details.

(A5) The map $\lambda : y \to$ {globally attracting set of $\dot{x}(t) \in h(x(t), y)$} is upper semi-continuous. $\dot{y}(t) \in G(y(t))$ has a globally attracting set, A_0, that is also Lyapunov stable. Here $G(y) := \overline{co}\left(\bigcup_{x\in\lambda(y)} g(x, y)\right)$. Further, the y iterates are track the aforementioned DI.

Main result

Almost surely the set of accumulation points is given by

$$\left\{(x, y) \mid \underline{\lim}_{n\to\infty} d\left((x, y), (x_n, y_n)\right) = 0\right\} \subseteq \bigcup_{y\in A_0} \{(x, y) \mid x \in \lambda(y)\}. \tag{4}$$

Application: The Lagrangian dual problem

- ✓ To solve the constrained minimization problem one often constructs an associated two timescale stochastic approximation algorithm.
- ✓ The analysis involves considering a family of minimum sets. Hitherto in literature these minimum sets are assumed to be singletons.
- ✓ We extend this analysis to the general case of set-valued minimum sets.

Stability of Stochastic Approximations with 'Controlled Markov' Noise and Temporal Difference Learning, [4]

- ✓ In [4] a 'stability theorem' for stochastic approximation (SA) algorithms with 'controlled Markov' noise. The iterates are shown to track a solution to a differential inclusion defined in terms of the ergodic occupation measures associated with the 'controlled Markov' process.
- ✓ We improve the general algorithm of Temporal Difference Learning using our framework.

References

[1] M. Benaïm. A dynamical system approach to stochastic approximations. *SIAM J. Control Optim.*, 34(2):437–472, 1996.

[2] J. Hofbauer M. Benaïm, S. Sorin. Stochastic approximations and differential inclusions. *SIAM Journal on Control and Optimization*, pages 328–348, 2005.

[3] Arunselvan Ramaswamy and Shalabh Bhatnagar. A generalization of the borkar-meyn theorem for stochastic recursive inclusions. *arXiv preprint arXiv:1502.01953*, 2015.

[4] Arunselvan Ramaswamy and Shalabh Bhatnagar. Stability of stochastic approximations withcontrolled markov'noise and temporal difference learning. *arXiv preprint arXiv:1504.06043*, 2015.

[5] Arunselvan Ramaswamy and Shalabh Bhatnagar. Stochastic recursive inclusion in two timescales with an application to the lagrangian dual problem. *arXiv preprint arXiv:1502.01956*, 2015.

[6] Arunselvan Ramaswamy and Shalabh Bhatnagar. Gradient based learning algorithms with constant-error gradient estimators. *arXiv preprint arXiv:1604.00151*, 2016.