Optimal Auctions for Two goods with Uniformly Distributed Valuations

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Optimal Auctions

An auction mechanism that generates the highest expected revenue to the seller.

The setup

- Auctioneer must sell a single item to a single buyer.
- The buyer has a valuation z known only to him.
- z is picked from a distribution f that is known both to the buyer and the seller.
- The auction must be designed so that the buyer reports his valuation truthfully.
- Also, the buyer must NOT be asked to pay more than z.

Optimization problem

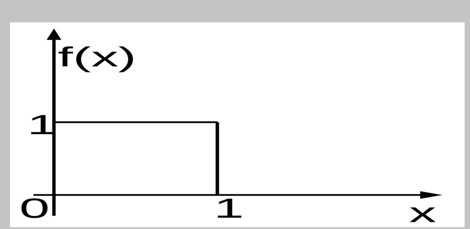
- **Objective:** Design an allocation function $q : \mathbb{R}_+ \to [0,1]$, and a payment function $t : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\mathbb{E}_{z\sim f} t(z)$ is maximized.
- **Constraints:** (1) Buyer must report z truthfully. (2) Buyer must not be asked to pay more than z.

$$\max_{q(\cdot),t(\cdot)} \mathbb{E}_{z\sim f} t(z)$$
$$\text{subject to IC and IR constraints.}$$

Myerson's Optimal solution

- Define the virtual valuation function
$$\phi(z) := z - \frac{1-F(z)}{f(z)}.$$
- Assume $\phi(z)$ increases in z.
- The solution is as follows:
$$(q(z), t(z)) = \begin{cases} (0,0) & \text{if } z \le \phi^{-1}(0), \\ (1, \phi^{-1}(0)) & \text{if } z > \phi^{-1}(0). \end{cases}$$

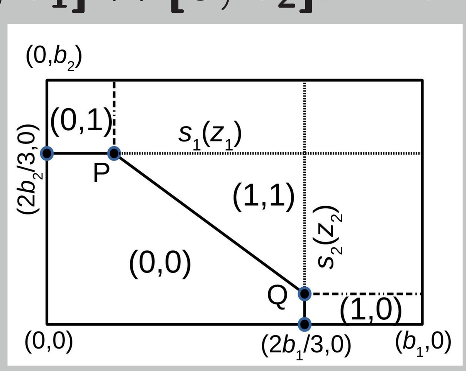
An Example



- Let $z \sim \text{unif}[0,1]$. $F(z) = z$, $f(z) = 1$, and thus $\phi(z) = z - (1-z) = 2z - 1$.
- $\phi(z) = 0$ when $z = 1/2$.
- So buyer gets the item for $1/2$, if his valuation is at least $1/2$. He doesn't get the item if his valuation is not even $1/2$.
- Optimal auction is a *take-it-or-leave-it* offer for a reserve price. The reserve price depends only on f.

Two-item Optimal auctions

- The optimal auction when the auctioneer has two items to sell, is NOT Myerson's auction repeated twice.
- Consider $z \sim \text{unif}[0, b_1] \times [0, b_2]$. The optimal auction is:



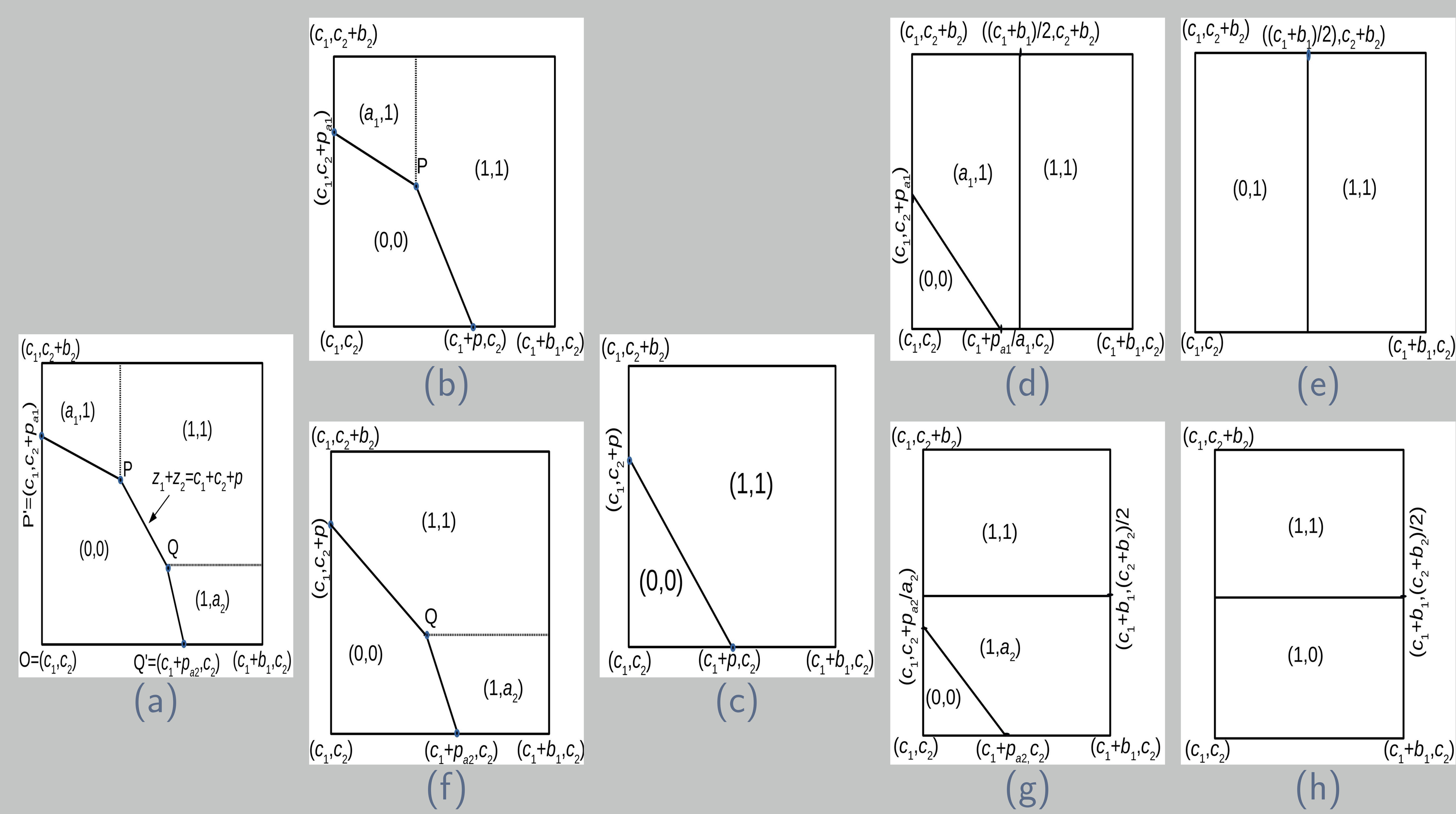
where line joining P, Q is
$z_1 + z_2 = (2b_1 + 2b_2 - \sqrt{2b_1 b_2})/3$.
- Bundling plays a crucial role in determining the optimal auction.

Our contribution

- General problem of two-item optimal auctions remains unsolved. It is unsolved even when f is restricted to uniform distribution over arbitrary rectangles in $\mathbb{R}^2_+$.
- We solve the problem when $z \sim \text{unif}[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$.
- The solution when $(c_1, c_2) = (0,0)$ was solved by forming a dual problem, which is an optimal transport problem that transfers mass from D to itself, subject to the constraint that the difference between the mass densities before and after the transfer equals $\bar{\mu}$, a signed measure defined by f.
- We solve the optimal transport problem so that the difference *strictly convex dominates* $\bar{\mu}$, for the uniform distribution on arbitrary rectangles.

Theorem 1

Let $z \sim \text{unif}[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$, with $(c_1, c_2, b_1, b_2) \ge \mathbf{0}$. The structure of the optimal solution takes one of the following eight structures.



Theorem 2

1. The structure of the optimal solution is one of the four structures depicted in Figures (a),(b),(f) and (c), whenever (c_1, c_2, b_1, b_2) belong to the set
$$\left\{c_1 \le b_1, c_2 \le 2b_2 \frac{b_1 + c_1}{b_1 + 3c_1}\right\} \cup \left\{c_2 \le b_2, c_1 \le 2b_1 \frac{b_2 + c_2}{b_2 + 3c_2}\right\}.$$
2. The structure of the optimal solution is one of the three depicted in Figures (c),(d), and (e), whenever $c_1 \le b_1$ and $c_2 \ge 2b_2 \left(\frac{b_1+c_1}{b_1+3c_1}\right)$. Specifically, the structure is that depicted in Figure (e), if $c_2 \ge 2b_2(b_1/(b_1 - c_1))^2$.
3. The structure of the optimal solution is one of the three depicted in Figures (c),(g), and (h), whenever $c_2 \le b_2$ and $c_1 \ge 2b_1 \left(\frac{b_2+c_2}{b_2+3c_2}\right)$. Specifically, the structure is that depicted in Figure (h), if $c_1 \ge 2b_1(b_2/(b_2 - c_2))^2$.
4. The structure is that depicted in Figure (c), whenever $c_1 \ge b_1$ and $c_2 \ge b_2$.

Interpreting the solutions

1. When both c_1 and c_2 are small, the solution is very close to the case $(c_1, c_2) = (0,0)$. The difference is that the buyer gets item 1 with some positive probability, even when z_1 is low. Similar is the case for item 2.
2. When c_1 is small and c_2 is sufficiently large, the optimal mechanism is to simply sell the second item with probability 1 for the least valuation c_2, and implement the Myerson's auction for item 1. Similar is the case when c_2 is small and c_1 is sufficiently large.
3. When both c_1 and c_2 are large, the optimal mechanism is to bundle the two goods and sell the bundle at the reserve price.
4. Below is the phase diagram indicating the optimal menus when $b_1 = 2$ and $b_2 = 1$.