Separable Convex Optimization with Linear Ascending Constraints

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Introduction

We consider the following separable convex optimization problem with linear inequality constraints. The problem arises in a wide variety of resource allocation settings. Let $w_i : \mathbb{R} \to \mathbb{R}, i = 1, 2, \cdots, n$ be strictly concave and continuously differentiable functions. We wish to minimize a separable objective function $W : \mathbb{R}^n \to \mathbb{R}$ as in

$$\text{Problem } \Pi :$$

$$\begin{aligned} \text{Maximize } \; & W(x) := \sum_{i=1}^{n} w_i(x_i) \\ \text{subject to } \; & x_i \geq 0 \quad i = 1, 2, \cdots, n, \\ & x_1 \leq \alpha_1, \\ & x_1 + x_2 \leq \alpha_1 + \alpha_2, \\ & \vdots \quad \leq \quad \vdots \\ & x_1 + x_2 + \cdots + x_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \end{aligned}$$

- $\alpha_i \geq 0$ for $i = 1, 2, \cdots, n$.
- The constraint set of problem Π is called Linear ascending constraints. Let F denote the constraint set.

Inventory Management

We give a simple example in inventory management to motivate the need to study problem Π.

- A single good produced at n time instants.
  - x_i is the quantity of good produced at i^{th} instant.
  - $w_i(x_i)$ is the convex cost function.
  - α_i is the consumption at i^{th} instant.
  - Demand has to be met at each instant.

$$\begin{aligned} \text{At time } 1: \quad & x_1 \geq \alpha_1 \\ \text{At time } 2: \quad & x_1 - \alpha_1 + x_2 \geq \alpha_2. \\ \implies & x_1 + x_2 \geq \alpha_1 + \alpha_2. \\ & \vdots \\ \text{At time } n: \quad & x_1 + x_2 + \cdots + x_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \end{aligned}$$

- Minimize total production cost : $\sum_{i=1}^{n} w_i(x_i)$

In addition to the applications in operations research, problem Π arises in certain resource allocation problems in wireless communications.

The algorithms that solve problem Π fall under two broad categories.

- Greedy algorithms: At each step, the variable that gives the largest increase in objective function is incremented.
- Decomposition algorithms: Decomposes the optimization problem into simpler sub problems.

We now give a decomposition algorithm that solves problem Π.

A Decomposition Algorithm

- Decompose problem Π into single sum constraint problems.

$$\text{SubProblem}(l, u) : \text{maximize} \sum_{i=l+1}^{u} w_i(x_i) \qquad \sum_{i=l+1}^{u} x_i = \sum_{i=l+1}^{u} \alpha_i.$$

- There exists indices $0 = s(0) < s(1) < s(2) < \cdots < s(p) = n$ such that the solution to Sub problem$(s(k), s(k+1)), k = 0, 1, \cdots, p-1$ gives the solution to problem Π.
- Complexity
  - Complexity to find $s(1), s(2), \cdots, s(p)$ is $\mathcal{O}(n^2)$.
  - Complexity of the algorithm is atleast $\mathcal{O}(n^2)$.
  - The complexity of the fastest known algorithm that solves problem Π is $\mathcal{O}(n \log n)$.

Network Utility Maximization

Network Structure

- A network with a special structure.

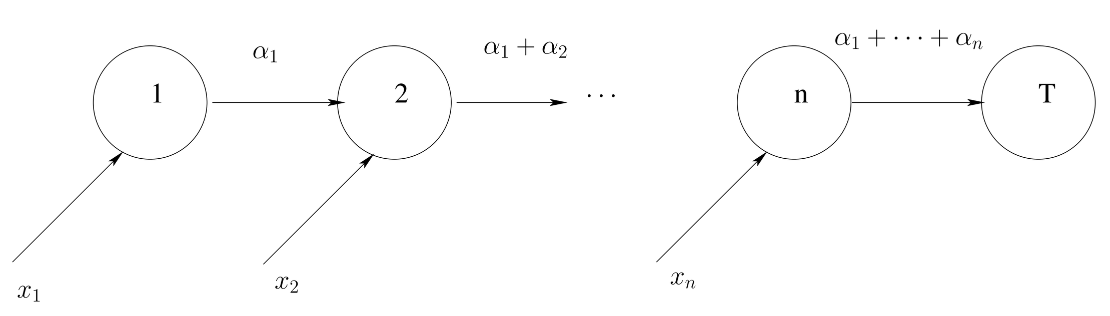

Figure 1: Network structure

- n users sending data across the network.
- User i derives a utility $w_i(x_i)$.
- Maximize sum utility subject to the flow constraints of the network.

$$\text{System}((x_i); W, F) : \; \text{Maximize} \; \sum_{i=1}^{n} w_i(x_i) \qquad (x_i) \in F.$$

- $w_i, i = 1, 2, \cdots, n$ are strictly concave, increasing and continuously differentiable functions.

Kelly Decomposition

- Distributed Optimization
  - The network does not know the utility functions.
  - Users do not know the network structure.
- Kelly decomposition (Kelly et al.)
  - Decomposes the system problem into n user problems and a network problem.

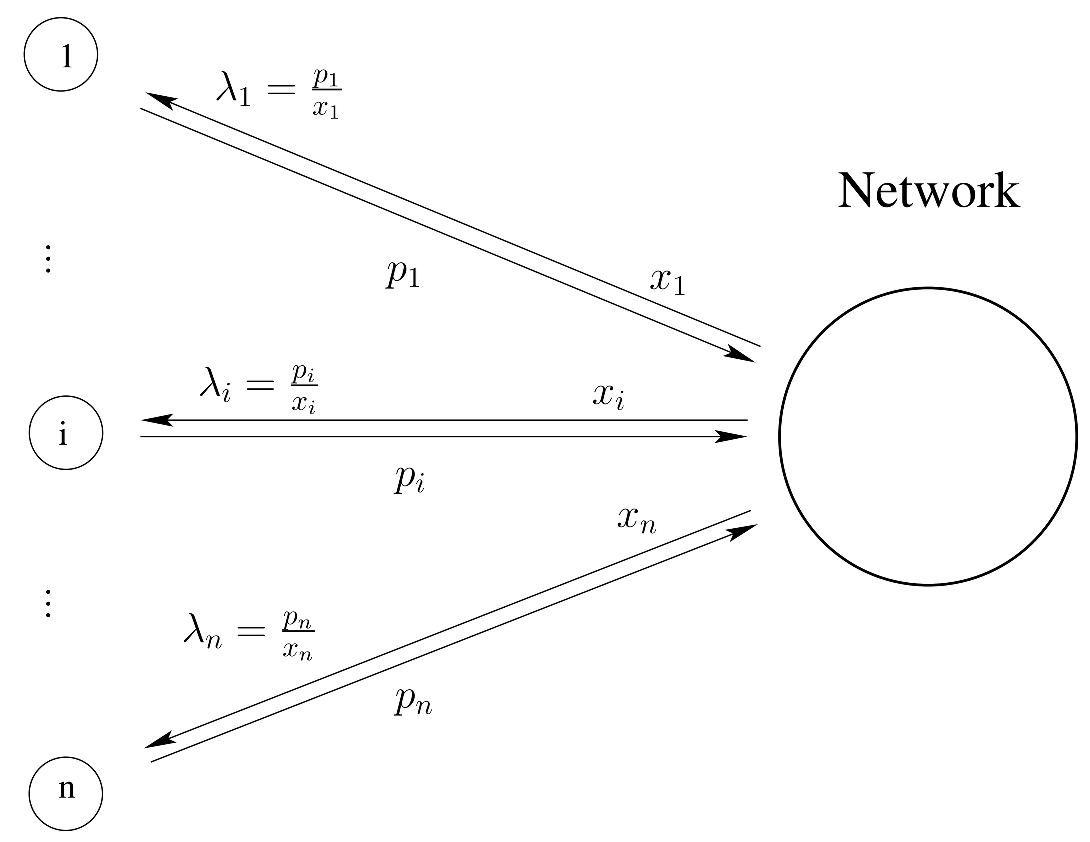

Figure 2: Kelly Decomposition

- The network sets λ_i, the cost per unit rate for user i.
- Let p_i be the amount user i is willing to pay.
- User i expects a rate $\frac{p_i}{\lambda_i}$.
- Choose p_i to maximize the net utility of user i.

$$\text{User}(p_i; \lambda_i) : \quad \text{Maximize} \quad w_i\left(\frac{p_i}{\lambda_i}\right) - p_i \qquad p_i \geq 0.$$

- Based on (p_i), the network allocates rates in a proportionally fair manner.

$$\text{Network}((x_i); (p_i), F) : \quad \text{Maximize} \quad \sum_{i=1}^{n} p_i \cdot \log x_i \qquad (x_i) \in F.$$

- Let (x_i) maximize the network problem.
- The updated cost per unit rate, $\lambda_i = \frac{p_i}{x_i}$.

Algorithm

- An iterative method

$$(\lambda_i^{(0)}) \to (p_i^{(0)}) \to \underbrace{(x_i^{(0)}) \to \left(\lambda_i^{(1)} = \frac{p_i^{(0)}}{x_i^{(0)}}\right) \to (p_i^{(1)}) \to (\hat{x}_i)}_{\text{T(x)}}$$

- The optimal solution to the system problem, $x^\star$, satisfies

$$x^\star = T(x^\star).$$

- $T(x)$ has multiple fixed points.
- Rate update

$$x^{(k+1)} = (1 - a(k)) \cdot x^{(k)} + a(k) \cdot T(x^{(k)}).$$

- Take $a(k) = \frac{1}{k}$.

Proof of Convergence

- $x^{(k)}$ approximates the trajectory of the following ODE.

$$\dot{x}(t) = T(x) - x.$$

- The equilibrium points of the ODE are the fixed points of $T(x)$.
- $x(t)$ converges to an equilibrium point shown via Lyapunov theory.
- $W(x)$ is the Lyapunov function.
- $x(t) \to x^\star$, hence $x^{(k)} \to x^\star$.

A Geometric Solution to the Network Problem

- Consider the network problem for the case $n = 4$.
- Let t_0 be the origin and t_1, t_2, t_3, t_4 be points in $\mathbb{R}^2$ with t_i located at $(\sum_{j=1}^{i} \alpha_j, \sum_{j=1}^{i} p_j)$.
- The solution to the network problem is obtained from the 'concave cover' of the points t_1, t_2, t_3 and t_4.
- See figure 3. The piece-wise linear function formed by the line segments $t_0 - t_1$, $t_1 - t_3$ and $t_3 - t_4$ is the concave cover of the points t_1, t_2, t_3 and t_4.

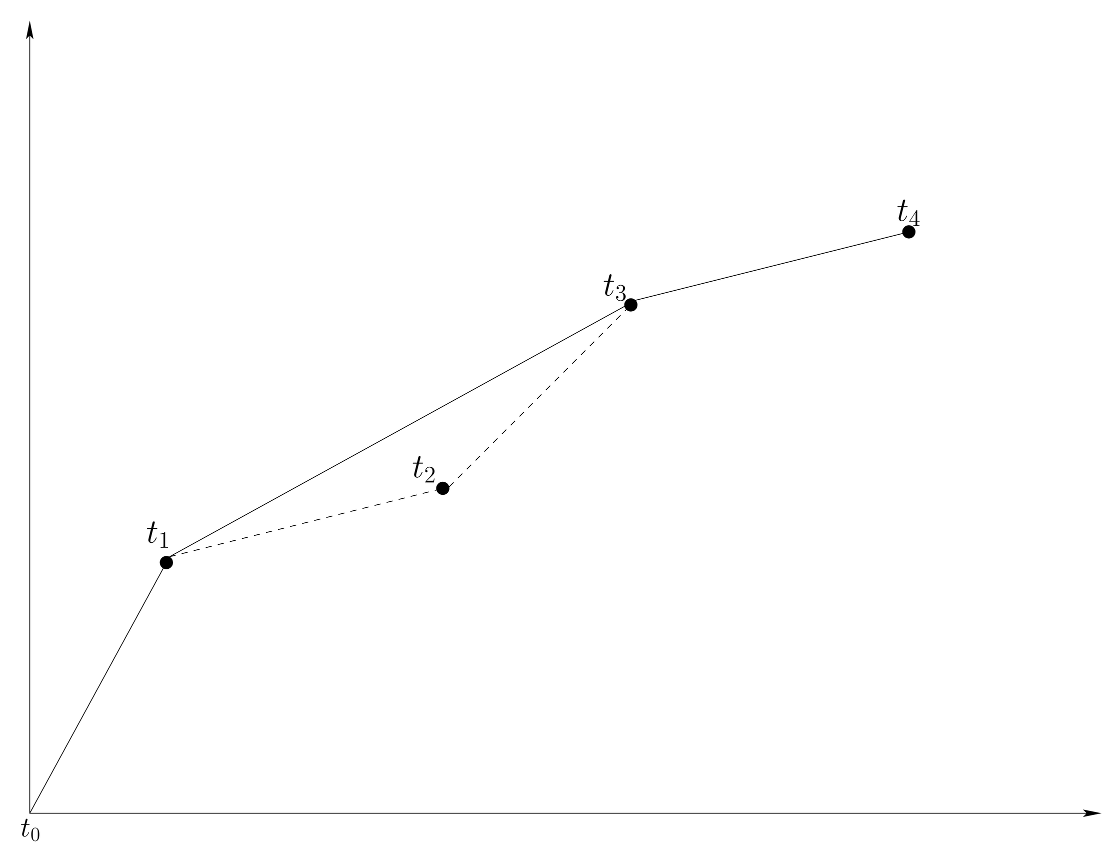

Figure 3: The concave cover

- $\lambda_1 =$ Slope of $t_0 - t_1$,
  $\lambda_2 = \lambda_3 =$ Slope of $t_1 - t_3$,
  $\lambda_4 =$ Slope of $t_3 - t_4$.
- $x_i = \frac{p_i}{\lambda_i}, i = 1, 2, 3$ and 4.
- String algorithm (Muckstadt et al.) finds the concave cover in $\mathcal{O}(n)$ steps.

Conclusion

In this work, we first proposed a centralized algorithm that solves problem Π. The algorithm solves the optimization problem by decomposing it into single constraint problems. The complexity of the decomposition technique is atleast $\mathcal{O}(n^2)$. Faster algorithms exist that solves problem Π in $\mathcal{O}(n \log n)$ steps. In the second part, we solve problem Π in a distributed setting using Kelly decomposition. The distributed algorithm is efficient due to String algorithm that solves the network problem in $\mathcal{O}(n)$ steps. The algorithm compares favorably to the algorithm of Kelly-Mauloo-Tan (KMT) in certain aspects. Although KMT uses Kelly decomposition, it does not solve the network problem in each step. As a result, the rates allocated at intermediate steps may not be feasible. This may also result in a slower convergence of the KMT algorithm.

References

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