On the Communication Complexity for SK Generation in the Multiterminal Source Model

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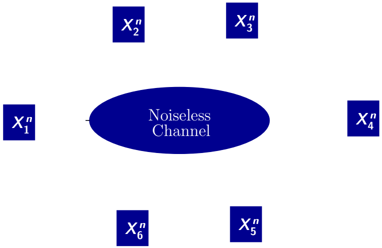
Outline

1. The Multiterminal Source Model
2. Lower Bound on R_{SK}
3. R_{SK}-maximal Sources

Preliminaries



- A set of terminals $\mathcal{M} = \{1, 2, \ldots, m\}$ wants to generate a group secret key.
- Each terminal has a component of a discrete memoryless multiple source, X_i^n, $\forall 1 \leq i \leq m$.

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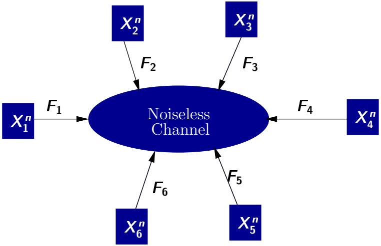

- Interactive communication is allowed among the terminals.
- $\mathbf{F} = \{F_1, F_2, \cdots, F_r\}$ is the *interactive communication* taking values in $\mathcal{F}$.
  Here F_j sent by some terminal i is a function of X_i^n and all the previous communication.
- Communication rate $= \frac{1}{n} \log |\mathcal{F}|$.
- After the communication the terminals compute a group secret key (SK) $\mathbf{K}^{(n)} = \mathbf{K}^{(n)}(X_{\mathcal{M}}^n)$.

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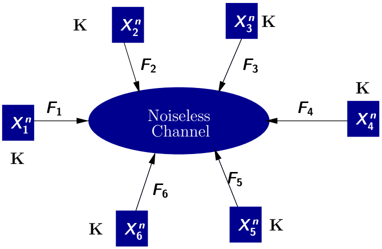

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Preliminaries(contd.)

- The secret key $\mathbf{K}^{(n)}$ satisfies the following property: for any $\epsilon > 0$ and for all sufficiently large n,
    - $\exists$ some function $g_i^{(n)}(X_i^n, \mathbf{F})$ such that $\Pr(\mathbf{K}^{(n)} \neq g_i^{(n)}(X_i^n, \mathbf{F})) \leq \epsilon, \forall 1 \leq i \leq m$. (*Recoverability*)
    - $I(\mathbf{K}^{(n)}; \mathbf{F}) \leq \epsilon$ (*Strong secrecy*)
    - $\log|\mathcal{K}^{(n)}| - H(\mathbf{K}^{(n)}) \leq \epsilon$, where $\mathcal{K}^{(n)}$ is the range of $\mathbf{K}^{(n)}$. (*Uniformity*)
- If $\frac{1}{n}H(\mathbf{K}^{(n)}) \rightarrow R$ as $n \rightarrow \infty$, then R is an achievable secret key rate.
- Secret key capacity $\mathcal{C}(\mathcal{M}) = \sup R$.

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Evaluating SK Capacity

- $\mathcal{C}(\mathcal{M}) = H(X_{\mathcal{M}}) - \min_{(R_1,R_2,\ldots,R_m)\in\mathcal{R}_{\mathrm{CO}}} \sum_{i=1}^{m} R_i$.
[Csiszár & Narayan, 2004]

  - Here
  $$\mathcal{R}_{\mathrm{CO}} = \Big\{(R_1, R_2, \ldots, R_m) : R_i \geq 0, \forall 1 \leq i \leq m, \sum_{j\in B} R_j \geq H(X_B|X_{B^c}), \forall B \subsetneq \mathcal{M}, B \neq \phi\Big\}$$
  is the achievable communication rate region for all terminals to recover $\mathbf{X}_{\mathcal{M}}^n$.
  - $R_{\mathrm{CO}} = \min_{(R_1,R_2,\ldots,R_m)\in\mathcal{R}_{\mathrm{CO}}} \sum_{i=1}^{m} R_i$ is called the minimum rate of communication for omniscience.

Evaluating SK Capacity

- $\mathcal{C}(\mathcal{M}) = \min_{\mathcal{P}} \Delta(\mathcal{P})$. [Chan & Zheng, 2010]

  - $\Delta(\mathcal{P}) = \frac{1}{\ell-1}\left[H(X_{A_1}) + H(X_{A_2}) + \cdots + H(X_{A_\ell}) - H(X_{\mathcal{M}})\right]$.

  - Here $\mathcal{P} = \{A_1, A_2, \ldots, A_\ell\}$, $\ell \geq 2$, is a partition of $\mathcal{M}$ and $X_A = (X_i : i \in A)$.

  - The quantity $\min_{\mathcal{P}} \Delta(\mathcal{P})$ is called multipartite information.

    Observe for the case of $m = 2$ the quantity $\min_{\mathcal{P}} \Delta(\mathcal{P})$ equals $I(X_1; X_2)$.

  - There exists a unique finest partition $\mathcal{P}^*$ of $\mathcal{M}$ which minimizes $\Delta(\mathcal{P})$.

    We shall refer to $\mathcal{P}^*$ as the fundamental partition.

Communication Complexity

- R_{SK} = Communication complexity,
  is the minimum rate of communication required to achieve SK capacity.
- $R_{\mathrm{SK}} \leq R_{\mathrm{CO}}$. [Csiszár & Narayan, 2004]
- If $R_{\mathrm{SK}} = R_{\mathrm{CO}}$, we call the source R_{SK}-maximal.
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Lower Bound on Communication Complexity

Theorem (Mukherjee & Kashyap, '16)

$R_{SK} \geq CI(X_{\mathcal{M}}) - \mathrm{I}(X_{\mathcal{M}})$.

- The result is an extension of Tyagi's earlier work for two terminals, i.e., $m = 2$.
- $\mathrm{CI}(X_{\mathcal{M}})$ is the minimum rate of interactive common information.
- Fact: $H(X_{\mathcal{M}}) \geq \mathrm{CI}(X_{\mathcal{M}}) \geq \mathrm{I}(X_{\mathcal{M}})$

  and hence the lower bound is non-negative.
- $\mathrm{CI}(X_{\mathcal{M}})$ is difficult to compute in general.

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Evaluating $\mathrm{CI}(X_{\mathcal{M}})$: The Hypergraphical Source

- Consider a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.
- $\mathcal{V} = \mathcal{M}$.
- Associate with each hyperedge $e \in \mathcal{E}$ an i.i.d. sequence of n Bernoulli $(1/2)$ random variables ξ_e^n.
- Random variables associated with distinct hyperedges in $\mathcal{E}$ are independent.
- Define a multiterminal source as follows: $X_i^n = (\xi_e^n : e \in \mathcal{E} \text{ such that } i \in e)$.
- The multiterminal source $X_{\mathcal{M}}^n$ is known as the hypergraphical source.

Theorem

For a hypergraphical source we have $\mathrm{CI}(X_{\mathcal{M}}) = |\mathcal{E}_{\mathcal{P}^*}|$, where $\mathcal{E}_{\mathcal{P}^*}$ is the set of hyperedges intersecting with at least two parts of $\mathcal{P}^*$.

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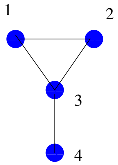

- Consider the following hypergraphical model.
  - $m = 4$ and $\mathcal{E} = \{\{1,2\},\{1,3\},\{2,3\},\{3,4\}\}$.
  - The random variables $(\xi_e)_{e\in\mathcal{E}}$ are Bernoulli (1/2) random variables.
  - $\mathcal{P}^* = \{\{1,2,3\},\{4\}\}$ and $\mathrm{I}(X_{\mathcal{M}}) = 1$.

- Therefore, $\mathrm{CI}(X_{\mathcal{M}}) = 1$ and hence, $\mathrm{CI}(X_{\mathcal{M}}) - \mathrm{I}(X_{\mathcal{M}}) = 0$.
- However, $R_{\mathrm{SK}} > 0$ as (X_1, X_2) is independent of X_4.

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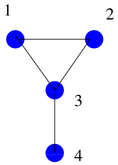

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  - $\mathcal{P}^* = \{\{1,2,3\},\{4\}\}$ and $\mathsf{I}(X_{\mathcal{M}}) = 1$.
- Therefore, $\mathsf{CI}(X_{\mathcal{M}}) = 1$ and hence, $\mathsf{CI}(X_{\mathcal{M}}) - \mathsf{I}(X_{\mathcal{M}}) = 0$.
- However, $R_{\text{SK}} > 0$ as (X_1, X_2) is independent of X_4.

When is a Source R_{SK}-maximal?

Theorem

A multiterminal source $X_{\mathcal{M}}$ with fundamental partition $\mathcal{P}^*$ is R_{SK}-maximal if for all $A \in \mathcal{P}^*$ we have $H(X_A|X_{A^c}) = 0$.

Theorem

A hypergraphical source $\mathcal{H} = (\mathcal{M}, \mathcal{E})$ is R_{SK}-maximal iff $\mathcal{E} = \mathcal{E}_{\mathcal{P}^*}$.

- **Example:** Hypergraphical source defined on the complete t-uniform hypergraph $K_{m,t}$:

  $\mathcal{V} = \mathcal{M}$.
  $\mathcal{E}$ is the set of all t-subsets of $\mathcal{M}$.


Thank you