Optimal Auctions for Two goods with Uniformly Distributed Valuations

EECS Symposium 2016

Thirumulanathan D

Advised by Prof. Rajesh Sundaresan and Prof. Y. Narahari

Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore

Overview

Overview

Introduction to Auctions

- When does an auction happen?
  It happens when there are one or more agents vying for an item that is ready to be sold.

- What does designing an auction mean?
  Deciding who should be allocated the item(s) and how much they pay. Mathematically, it is the design of two functions: the allocation function q and the payment function t.

- What is an *optimal* auction?
  It is an auction mechanism that generates the highest expected revenue to the seller.

The setup

Consider the simple case of an auctioneer selling a single item to a single buyer. We have the following assumptions:

- The buyer has a valuation z for the item known only to him.
- z is picked from a distribution f. The distribution is known to both the buyer and the seller.
- The auction must be designed so that the buyer reports his valuation truthfully.
- Also, the buyer must NOT be asked to pay more than z.
- Recall that the auction design involves designing the allocation function q, and the payment function t.

The setup

Consider the simple case of an auctioneer selling a single item to a single buyer. We have the following assumptions:

- The buyer has a valuation z for the item known only to him.
- z is picked from a distribution f. The distribution is known to both the buyer and the seller.
- The auction must be designed so that the buyer reports his valuation truthfully.
- Also, the buyer must NOT be asked to pay more than z.
- Recall that the auction design involves designing the allocation function q, and the payment function t.

The setup

Consider the simple case of an auctioneer selling a single item to a single buyer. We have the following assumptions:

- The buyer has a valuation z for the item known only to him.
- z is picked from a distribution f. The distribution is known to both the buyer and the seller.
- The auction must be designed so that the buyer reports his valuation truthfully.
- Also, the buyer must NOT be asked to pay more than z.
- Recall that the auction design involves designing the allocation function q, and the payment function t.

The setup

Consider the simple case of an auctioneer selling a single item to a single buyer. We have the following assumptions:

- The buyer has a valuation z for the item known only to him.
- z is picked from a distribution f. The distribution is known to both the buyer and the seller.
- The auction must be designed so that the buyer reports his valuation truthfully.
- Also, the buyer must NOT be asked to pay more than z.
- Recall that the auction design involves designing the allocation function q, and the payment function t.

The setup

Consider the simple case of an auctioneer selling a single item to a single buyer. We have the following assumptions:

- The buyer has a valuation z for the item known only to him.
- z is picked from a distribution f. The distribution is known to both the buyer and the seller.
- The auction must be designed so that the buyer reports his valuation truthfully.
- Also, the buyer must NOT be asked to pay more than z.
- Recall that the auction design involves designing the allocation function q, and the payment function t.

Single item Optimal Auctions

- Thus the objective of the seller is now to design an auction that solves the following optimization problem:

Maximize the expected revenue ($max_{q(\cdot),t(\cdot)} \mathbb{E}_{z\sim f}\, t(z)$)

subject to (1) Truthful Extraction of valuation
(2) Buyer is asked to pay at most his valuation

- Myerson [1981] solved this problem. Define the virtual valuation function $\phi(z) := z - \frac{1-F(z)}{f(z)}$. The solution is then given by

$$(q(z), t(z)) = \begin{cases} (0,0) & \text{if } z \leq \phi^{-1}(0), \\ (1, \phi^{-1}(0)) & \text{if } z > \phi^{-1}(0). \end{cases}$$

- The item is allocated if buyer's valuation is at least $\phi^{-1}(0)$, and he pays $\phi^{-1}(0)$. He is not allocated the item otherwise.

Single item Optimal Auctions

- Thus the objective of the seller is now to design an auction that solves the following optimization problem:

Maximize the expected revenue ($max_{q(\cdot),t(\cdot)} \mathbb{E}_{z \sim f}\, t(z)$)

subject to (1) Truthful Extraction of valuation
(2) Buyer is asked to pay at most his valuation

- Myerson [1981] solved this problem. Define the virtual valuation function $\phi(z) := z - \frac{1-F(z)}{f(z)}$. The solution is then given by

$$(q(z), t(z)) = \begin{cases} (0,0) & \text{if } z \leq \phi^{-1}(0), \\ (1, \phi^{-1}(0)) & \text{if } z > \phi^{-1}(0). \end{cases}$$

- The item is allocated if buyer's valuation is at least $\phi^{-1}(0)$, and he pays $\phi^{-1}(0)$. He is not allocated the item otherwise.

Single item Optimal Auctions

- Thus the objective of the seller is now to design an auction that solves the following optimization problem:

Maximize the expected revenue ($max_{q(\cdot),t(\cdot)}\mathbb{E}_{z\sim f}\, t(z)$)

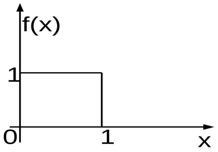
subject to (1) Truthful Extraction of valuation
(2) Buyer is asked to pay at most his valuation

- Myerson [1981] solved this problem. Define the virtual valuation function $\phi(z) := z - \frac{1-F(z)}{f(z)}$. The solution is then given by

$$(q(z), t(z)) = \begin{cases} (0,0) & \text{if } z \leq \phi^{-1}(0), \\ (1, \phi^{-1}(0)) & \text{if } z > \phi^{-1}(0). \end{cases}$$

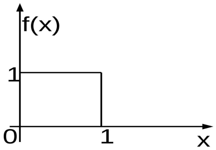
- The item is allocated if buyer's valuation is at least $\phi^{-1}(0)$, and he pays $\phi^{-1}(0)$. He is not allocated the item otherwise.

An Example



- $z \sim \text{unif}[0,1]$. $F(z) = z$, $f(z) = 1$, and thus $\phi(z) = z - (1 - z) = 2z - 1$.
- $\phi(z) = 0$ when $z = 1/2$. So, the buyer gets the item for $1/2$, if his valuation is at least $1/2$. He doesn't get it if his valuation is not even $1/2$.
- Observe that the optimal auction is a *take-it-or-leave-it* offer for a reserve price. The reserve price depends only on the distribution function f.

An Example

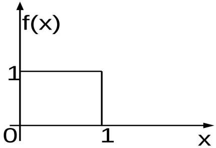


- $z \sim \text{unif}[0,1]$. $F(z) = z$, $f(z) = 1$, and thus $\phi(z) = z - (1 - z) = 2z - 1$.

- $\phi(z) = 0$ when $z = 1/2$. So, the buyer gets the item for $1/2$, if his valuation is at least $1/2$. He doesn't get it if his valuation is not even $1/2$.

- Observe that the optimal auction is a *take-it-or-leave-it* offer for a reserve price. The reserve price depends only on the distribution function f.

An Example

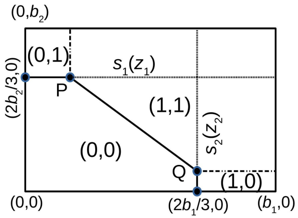


- $z \sim \text{unif}[0,1]$. $F(z) = z$, $f(z) = 1$, and thus $\phi(z) = z - (1 - z) = 2z - 1$.
- $\phi(z) = 0$ when $z = 1/2$. So, the buyer gets the item for $1/2$, if his valuation is at least $1/2$. He doesn't get it if his valuation is not even $1/2$.
- Observe that the optimal auction is a *take-it-or-leave-it* offer for a reserve price. The reserve price depends only on the distribution function f.

Overview

Two-item Optimal Auction

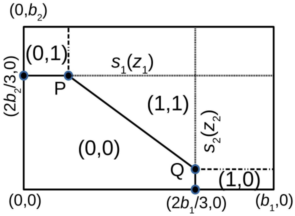
- The problem of optimal auction for one-item was solved in 1981. That for two-items is unsolved even today.
- Solutions for certain distributions are known, however. When $z \sim \text{unif}[0, b_1] \times [0, b_2]$, (Daskalakis et al. [2013]).



- So the buyer gets only item 1 if $\mathbf{z_1 > 2b_1/3}$ (and z_2 is small), only item 2 if $\mathbf{z_2 > 2b_2/3}$ (and z_1 is small), and both items if $\mathbf{z_1 + z_2 > (2b_1 + 2b_2 - \sqrt{2b_1b_2})/3}$.
- Bundling plays a crucial role in two-item optimal auctions.

Two-item Optimal Auction

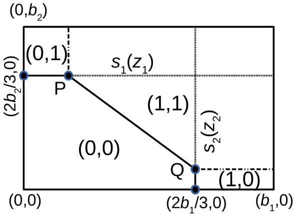
- The problem of optimal auction for one-item was solved in 1981. That for two-items is unsolved even today.
- Solutions for certain distributions are known, however. When $z \sim \text{unif}[0, b_1] \times [0, b_2]$, (Daskalakis et al. [2013]).



- So the buyer gets only item 1 if $\mathbf{z_1 > 2b_1/3}$ (and z_2 is small), only item 2 if $\mathbf{z_2 > 2b_2/3}$ (and z_1 is small), and both items if $\mathbf{z_1 + z_2 > (2b_1 + 2b_2 - \sqrt{2b_1b_2})/3}$.
- Bundling plays a crucial role in two-item optimal auctions.

Two-item Optimal Auction

- The problem of optimal auction for one-item was solved in 1981. That for two-items is unsolved even today.
- Solutions for certain distributions are known, however. When $z \sim \text{unif}[0, b_1] \times [0, b_2]$, (Daskalakis et al. [2013]).

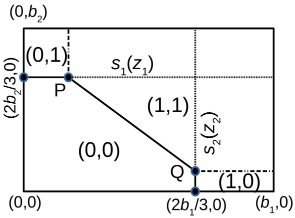


- So the buyer gets only item 1 if $\mathbf{z_1 > 2b_1/3}$ (and z_2 is small), only item 2 if $\mathbf{z_2 > 2b_2/3}$ (and z_1 is small), and both items if $\mathbf{z_1 + z_2 > (2b_1 + 2b_2 - \sqrt{2b_1 b_2})/3}$.
- Bundling plays a crucial role in two-item optimal auctions.

Two-item Optimal Auction

- The problem of optimal auction for one-item was solved in 1981. That for two-items is unsolved even today.

- Solutions for certain distributions are known, however. When $z \sim \text{unif}[0, b_1] \times [0, b_2]$, (Daskalakis et al. [2013]).



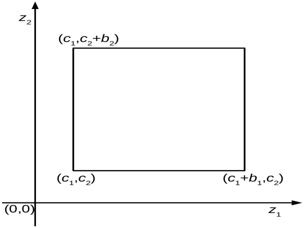
- So the buyer gets only item 1 if $\mathbf{z_1 > 2b_1/3}$ (and z_2 is small), only item 2 if $\mathbf{z_2 > 2b_2/3}$ (and z_1 is small), and both items if $\mathbf{z_1 + z_2 > (2b_1 + 2b_2 - \sqrt{2b_1b_2})/3}$.

- Bundling plays a crucial role in two-item optimal auctions.

Overview

Uniform Distribution on arbitrary rectangles

- Consider the buyer's valuations to be uniformly distributed in the intervals $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$ for arbitrary nonnegative values of c_1, c_2, b_1, b_2.



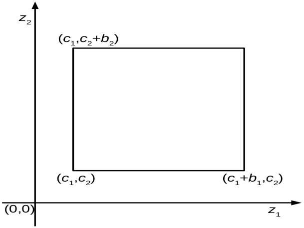
- In our work, we prove the following theorem:

Theorem

The structure of the optimal solution takes one of the following eight structures for any nonnegative (c_1, c_2, b_1, b_2).

Uniform Distribution on arbitrary rectangles

- Consider the buyer's valuations to be uniformly distributed in the intervals $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$ for arbitrary nonnegative values of c_1, c_2, b_1, b_2.

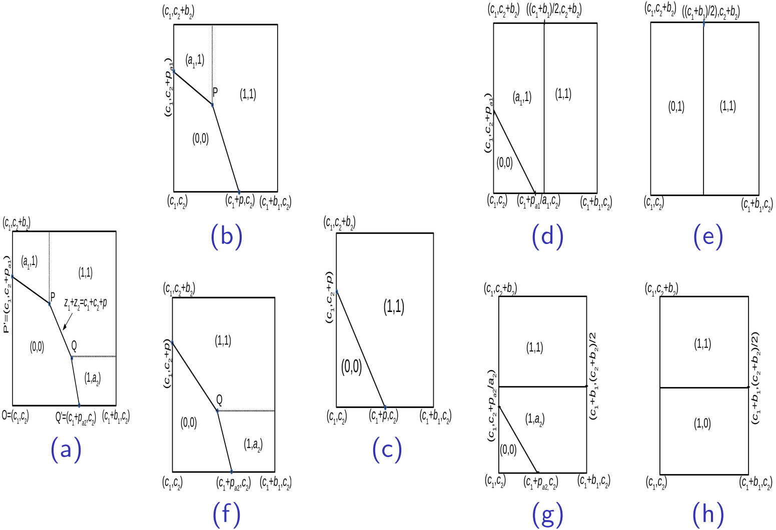


- In our work, we prove the following theorem:

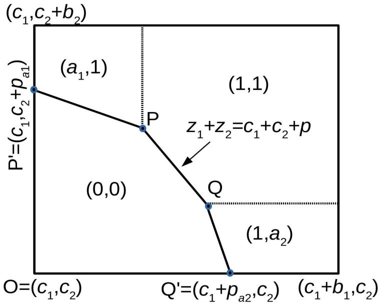
Theorem

The structure of the optimal solution takes one of the following eight structures for any nonnegative (c_1, c_2, b_1, b_2).

The structure of optimal auctions

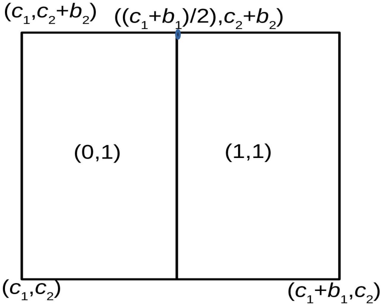


When both c_1 and c_2 are low



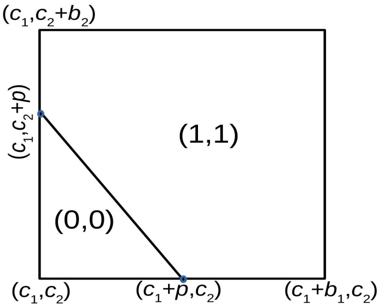
The solution is very close to the case $(c_1, c_2) = (0, 0)$. The difference is that the buyer gets item 1 with some positive probability, even when z_1 is low. Similar is the case for item 2.

When c_1 is low and c_2 is high, or vice-versa



- Since c_2 is very high, item 2 is allocated with probability 1 for the least possible price c_2, no matter what z_2 is. Myerson auction is conducted for item 1.
- Similar is the case when c_2 is low and c_1 is high.

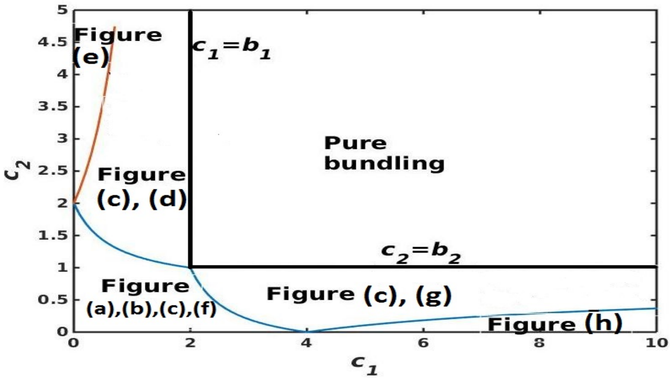
When both c_1 and c_2 are high



For higher values of c_1 and c_2, the optimal auction is a *take-it-or-leave-it* auction with a reserve price, with both the items bundled as a single item.

Phase diagram

The phase diagram indicates the optimal menu for all the values of c_1, c_2, when $b_1 = 2$ and $b_2 = 1$.



Overview

Summary

- The optimal auction for a single item is a simple take-it-or-leave-it auction at a reserve price. The reserve price depends on the distribution f.
- The optimal auction for two-item case is much more complicated. It is NOT the single item optimal auction repeated twice. Bundling increases the revenue.
- In our work, we prove that the optimal auction when the valuations are uniformly distributed in the rectangle $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$ is to sell the items according to one of the eight simple menus.
- The auctions resemble the single item optimal auctions when either of c_1 or c_2 is high.

Summary

- The optimal auction for a single item is a simple take-it-or-leave-it auction at a reserve price. The reserve price depends on the distribution f.
- The optimal auction for two-item case is much more complicated. It is NOT the single item optimal auction repeated twice. Bundling increases the revenue.
- In our work, we prove that the optimal auction when the valuations are uniformly distributed in the rectangle $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$ is to sell the items according to one of the eight simple menus.
- The auctions resemble the single item optimal auctions when either of c_1 or c_2 is high.

Summary

- The optimal auction for a single item is a simple take-it-or-leave-it auction at a reserve price. The reserve price depends on the distribution f.
- The optimal auction for two-item case is much more complicated. It is NOT the single item optimal auction repeated twice. Bundling increases the revenue.
- In our work, we prove that the optimal auction when the valuations are uniformly distributed in the rectangle $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$ is to sell the items according to one of the eight simple menus.
- The auctions resemble the single item optimal auctions when either of c_1 or c_2 is high.

Summary

- The optimal auction for a single item is a simple take-it-or-leave-it auction at a reserve price. The reserve price depends on the distribution f.
- The optimal auction for two-item case is much more complicated. It is NOT the single item optimal auction repeated twice. Bundling increases the revenue.
- In our work, we prove that the optimal auction when the valuations are uniformly distributed in the rectangle $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$ is to sell the items according to one of the eight simple menus.
- The auctions resemble the single item optimal auctions when either of c_1 or c_2 is high.