Phase Retrieval and Hilbert Integral Equations — Beyond Minimum Phase

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1. Overview

Phase retrieval: Reconstruct a signal from its magnitude spectrum.

Minimum-phase signals allow for exact phase retrieval. The Fourier transform of a minimum-phase signal is given by

$$X(e^{\mathrm{j}\omega}) = \frac{\prod_{i=1}^{M}(1-a_i e^{-\mathrm{j}\omega})}{\prod_{i=M+1}^{M+N}(1-a_i e^{-\mathrm{j}\omega})}, \; |a_i|<1.$$

The log-magnitude and phase of minimum phase signals satisfy the Hilbert integral equations:

$$\mathcal{H}^p\{\log|X|\} = \arg X.$$

- We show that there exist larger classes of signals, for which exact phase retrieval is possible.
- We generalize Hilbert integral equations to 2-D.
- We introduce four new classes of signals that are a generalization of the minimum-phase class and for which the Hilbert integral equation holds between their log-magnitude and phase spectra.

2. Parametric Extension to 2D

Signal model:

$$X(e^{\mathrm{j}\boldsymbol{\omega}}) = \frac{\prod_{i=1}^{M}(1-a_i e^{-\mathrm{j}\omega_1} - b_i e^{-\mathrm{j}\omega_2})^{q_i}}{\prod_{i=M+1}^{M+N}(1-a_i e^{-\mathrm{j}\omega_1} - b_i e^{-\mathrm{j}\omega_2})^{q_i}}$$

Problem statement: Reconstruct x from $|X|$.

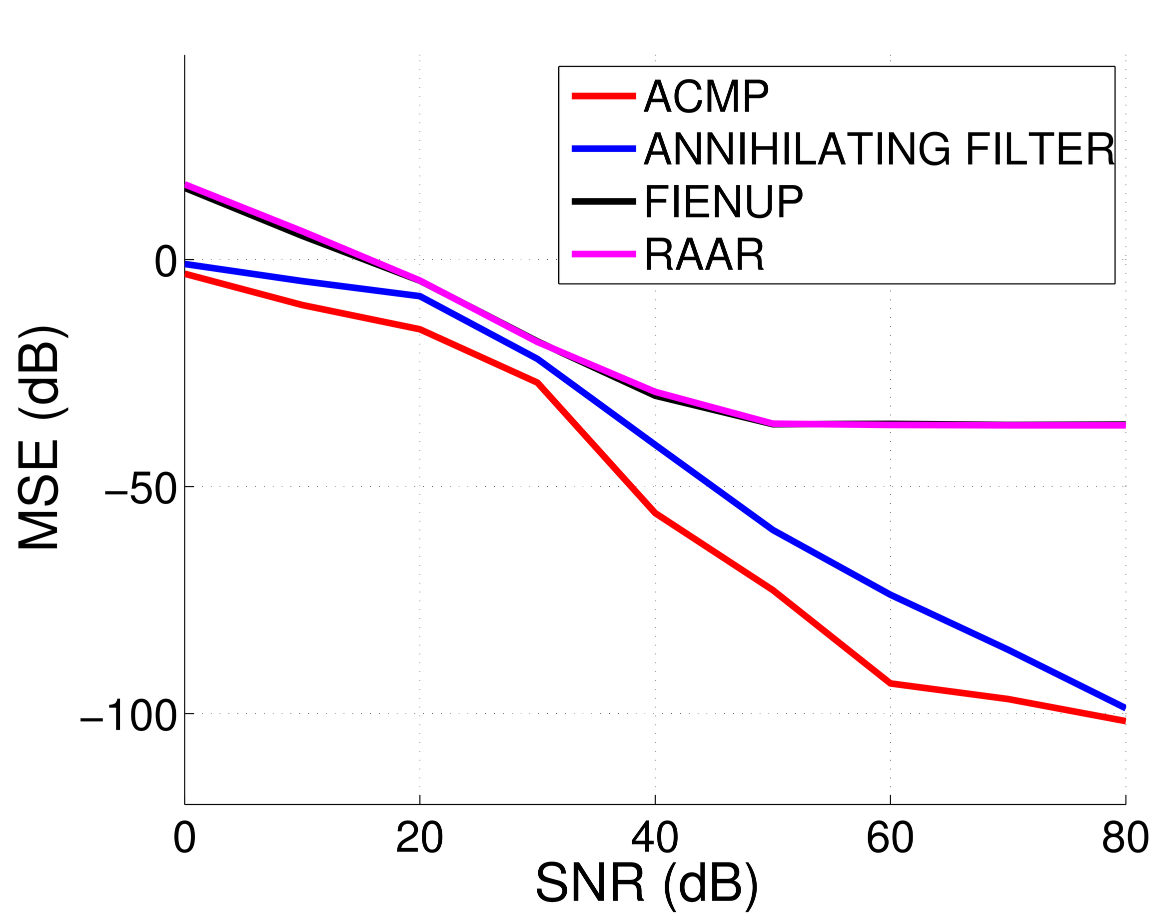
Proposed methodology:

$$-\frac{2(n_1+n_2)}{\binom{n_1+n_2}{n_1}}\left(\mathcal{F}^{-1}\{\log|X|\}u(n)\right) = \textstyle\sum_i p_i a_i^{n_1} b_i^{n_2}$$

where $p_i = q_i$ for $i \in [\![1, M]\!]$ and
$p_i = -q_i$, for $\in [\![M+1, M+N]\!]$

Parameter computation from sum-of-exponentials via the annihilating filter and algebraically-coupled matrix pencil method.

Performance comparisons:



3. Shift-Invariant Spaces

Signal model: $f(t) = \sum_{k\in\mathbb{Z}} c_k \phi(t-k)$

Problem statement: Given $|\hat{f}(\omega)|$, reconstruct $f(t)$.

$$\hat{f}(\omega) = \sum_{k\in\mathbb{Z}} c_k\, \hat{\phi}(\omega) e^{-\mathrm{j}k\omega} = \hat{\phi}(\omega) C(e^{\mathrm{j}\omega})$$

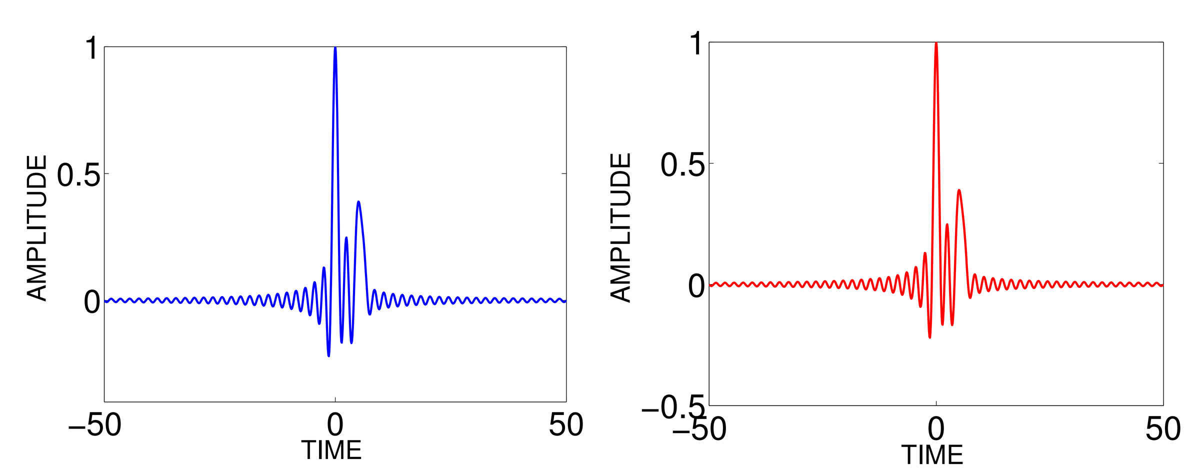
Since we know ϕ, the problem is reformulated: Given $|C(e^{\mathrm{j}\omega})|$, reconstruct $\{c_k\}_{k\in\mathbb{Z}}$

Causal delta dominant (CDD):
$c_k = \delta_k + a_k$, $\{a_k\}_{k\in\mathbb{Z}}$ is causal and $\|a_k\|_1 < 1$.

Exact reconstruction of f from $|\hat{f}(\omega)|$ if $\{c_k\}_{k\in\mathbb{Z}}$ is CDD.

$$c_k = \mathcal{F}^{-1}\left\{\exp\left(\mathcal{F}\{\mathcal{F}^{-1}\{(\log|C(e^{\mathrm{j}\omega})|^2)\}u_k\}\right)\right\}$$

Phase retrieval of bandlimited signals spanned by sinc kernel



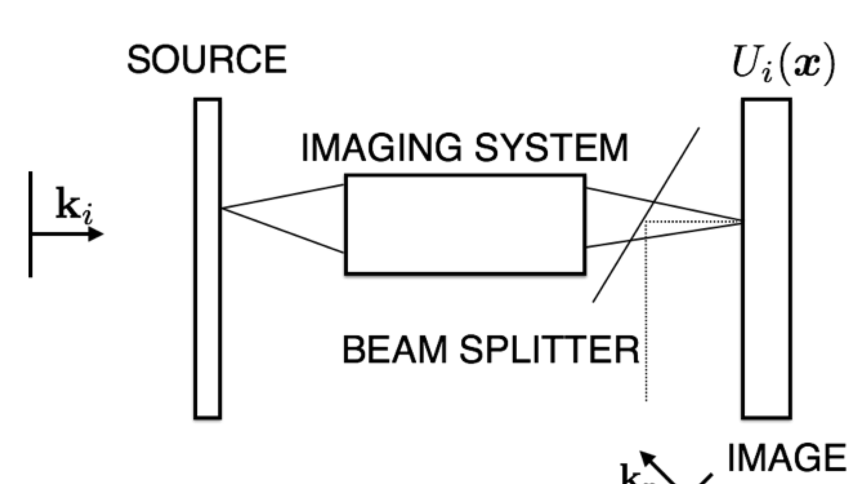
There exist CDD signal which are not minimum-phase and vice versa.

4. Phase Microscopy

We have measurements of the form

$$I(\boldsymbol{x}) = |U_i(\boldsymbol{x}) + U_r(\boldsymbol{x})|^2,$$

where $U_i(\boldsymbol{x})$ is the object wave and $U_r(\boldsymbol{x})$ is the reference wave.



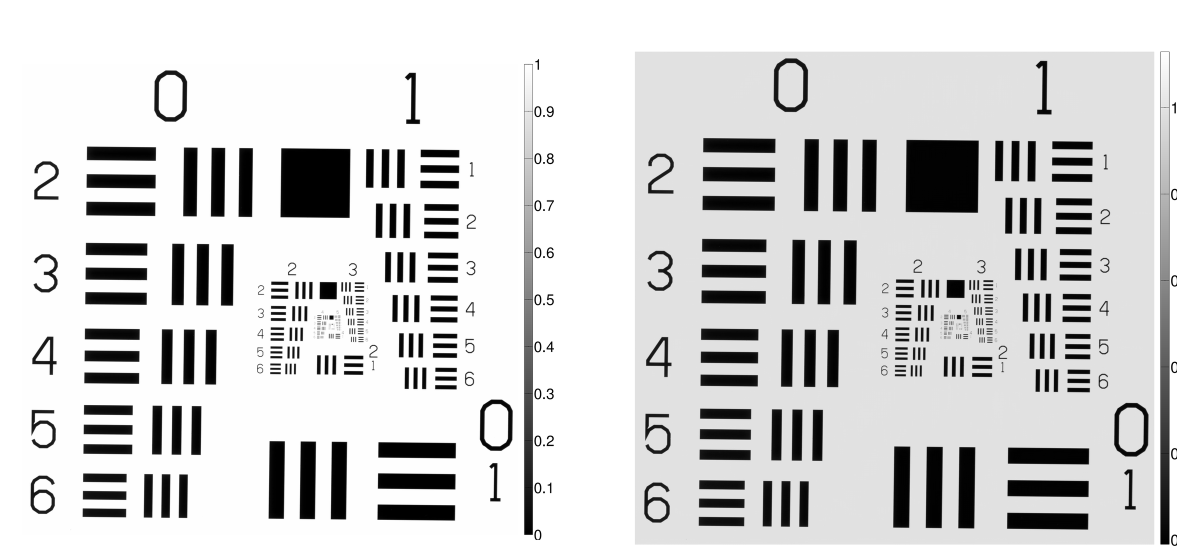
Problem statement: Reconstruct $U_i(\boldsymbol{x})$ from $I(\boldsymbol{x})$.

CDD structure:

$$\tilde{I}(\boldsymbol{x}) := \frac{I(\boldsymbol{x})}{|U_r(\boldsymbol{x})|^2} = |1+\tilde{U}_i(\boldsymbol{x})|^2,$$

provided that the reference wave intensity is larger than the object wave intensity, and reference wave frequency is larger than the spectral support of the object wave

Testing resolution accuracy on USAF target images



5. Hilbert Inegral Equations

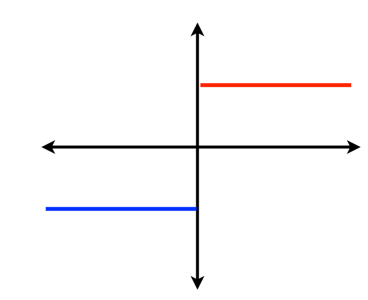
The real and imaginary parts of spectrum of causal signals form a Hilbert pair:

$$f(t) \text{ causal} \implies \hat{f}_r \overset{\mathcal{H}^c}{\longleftrightarrow} \hat{f}_i$$
$$x(n) \text{ causal} \implies X_r \overset{\mathcal{H}^p}{\longleftrightarrow} X_i$$

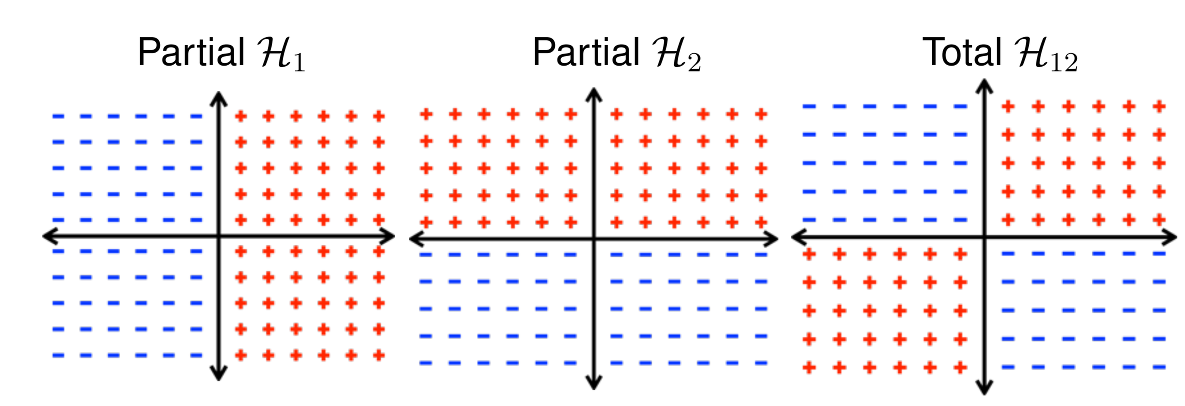
where

$$\mathcal{H}^c\{\hat{f}\} = \hat{f} * \frac{1}{\pi\omega}$$
$$\mathcal{H}^p\{X\} = X \circledast \frac{1}{2\pi}\cot(\omega/2)$$

The 1-D Hilbert transformation is a multiplication with the signum in the Fourier domain:



The 2-D extensions are direction-oriented:



For continuous-domain 2-D signals, we have

$$f(\boldsymbol{x}) \text{ first-quadrant} \implies \hat{f}_r \overset{\mathcal{H}_i^c}{\longleftrightarrow} \hat{f}_i$$

This result does not carry over to the discrete case.

We introduce the composite Hilbert transform: $\mathscr{H}^p\{X\} = X \circledast K$, where

$$K(e^{\mathrm{j}\boldsymbol{\omega}}) = \frac{1}{2}\left(\cot\left(\frac{\omega_1}{2}\right)\delta(\omega_2) + \cot\left(\frac{\omega_2}{2}\right)\delta(\omega_1)\right) + \cot\left(\frac{\omega_1}{2}\right) + \cot\left(\frac{\omega_2}{2}\right).$$

We now have:

$$x(\boldsymbol{n}) \text{ first quadrant} \implies X_r \overset{\mathscr{H}^p}{\longleftrightarrow} X_i$$

We introduce the following classes of *generalized minimum-phase* signals for which corresponding Hilbert relations hold between the log-magnitude and phase spectra.

$$X(e^{\mathrm{j}\boldsymbol{\omega}}) = \frac{\prod_{i=1}^{M}(1+Y_i(e^{\mathrm{j}\boldsymbol{\omega}}))}{\prod_{i=M+1}^{M+N}(1+Y_i(e^{\mathrm{j}\boldsymbol{\omega}}))}, |Y_i(e^{\mathrm{j}\boldsymbol{\omega}})|<1, \forall\boldsymbol{\omega}$$

$$\hat{f}(\boldsymbol{\omega}) = \frac{\prod_{i=1}^{M}(1+\hat{g}_i(\boldsymbol{\omega}))}{\prod_{i=M+1}^{M+N}(1+\hat{g}_i(\boldsymbol{\omega}))}, |\hat{g}_i(\boldsymbol{\omega})| \le \gamma < 1, \forall\boldsymbol{\omega}$$

References

[1] B. A. Shenoy and C. S. Seelamantula, "Exact phase retrieval for a class of 2-D parametric signals," IEEE Transactions on Signal Processing, vol. 63, no. 1, pp. 90-103, 2015.

[2] B. A. Shenoy, S. Mulleti, and C. S. Seelamantula, "Exact phase retrieval in principal shift-invariant spaces," IEEE Transactions on Signal Processing, vol. 64, no. 2, pp. 406-416, 2016.

[3] B. A. Shenoy, A. Anand, and C. S. Seelamantula, "Exact reconstruction in quantitative phase microscopy," Manuscript under preparation.

[4] B. A. Shenoy, S. Mulleti, and C. S. Seelamantula, "On Hilbert integral equations, generalized minimum-phase signals, and phase retrieval," Manuscript under revision, IEEE Transactions on Information Theory.

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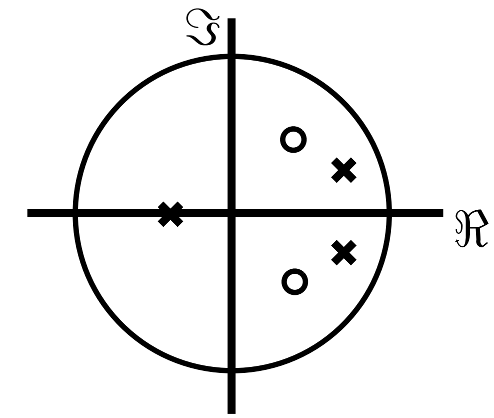
SPECTRUM LAB

Overview

Problem statement: Reconstruct a signal from its magnitude spectrum.

$$\text{Minimum-Phase: } X(e^{\mathrm{j}\omega}) = \frac{\prod_{i=1}^{M}(1 - a_i e^{-\mathrm{j}\omega})}{\prod_{i=M+1}^{M+N}(1 - a_i e^{-\mathrm{j}\omega})}, \ |a_i| < 1$$



$$\text{Hilbert Relations: } \log|X| \overset{\mathcal{H}^p}{\longleftrightarrow} \arg X \qquad \mathcal{H}^{\mathrm{p}}\{Y\} = \frac{1}{2\pi} Y * \cot(\omega/2)$$

Extensions:

- 2-D parametric extension:
  - Phase retrieval via parameter computation from a sum-of-exponentials signal.
- Signals in shift-invariant spaces:
  - We introduce the notion of causal delta dominant (CDD) signals
  - Demonstrate exact phase retrieval.
- Phase microscopy:
  - We show applicability of the CDD model to signals encountered in phase microscopy.
- Hilbert integral equations:
  - We extend the Hilbert relations to 2-D first-quadrant signals.
  - We introduce new classes of signals that are a generalization of the minimum-phase class.

2-D Parametric Extension

Minimum-Phase: $X(e^{\mathrm{j}\omega}) = \dfrac{\prod_{i=1}^{M}(1 - a_i e^{-\mathrm{j}\omega})}{\prod_{i=M+1}^{M+N}(1 - a_i e^{-\mathrm{j}\omega})}, \; |a_i| < 1$

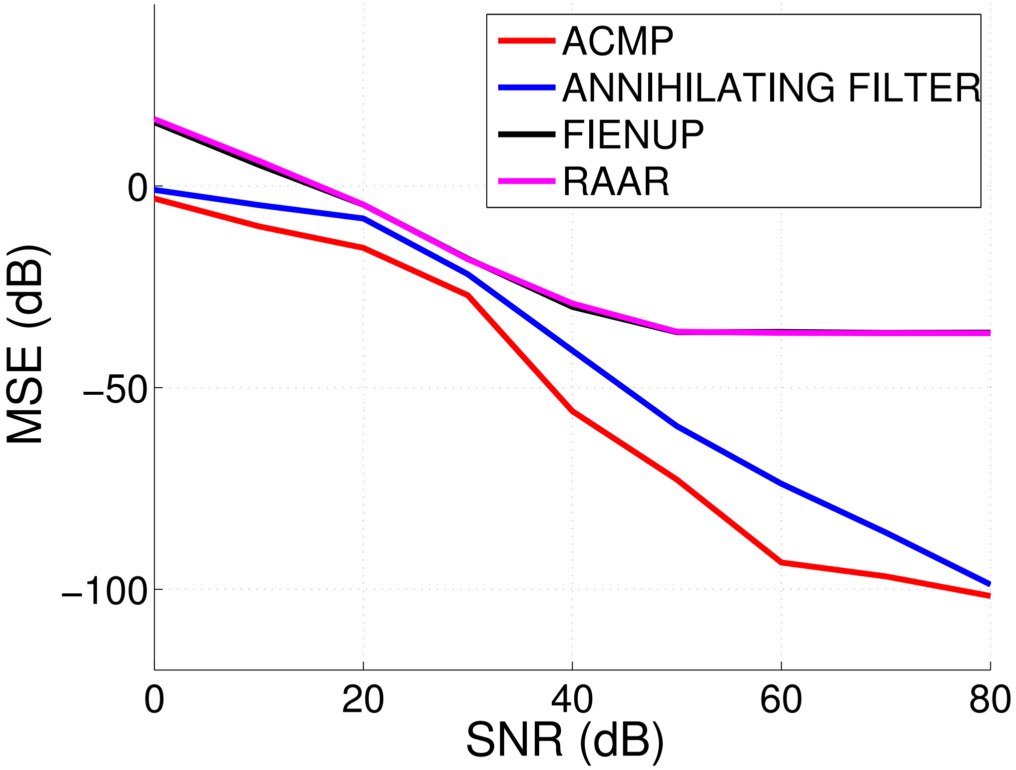
Extension: $X(e^{\mathrm{j}\boldsymbol{\omega}}) = \dfrac{\prod_{i=1}^{M}(1 - a_i e^{-\mathrm{j}\omega_1} - b_i e^{-\mathrm{j}\omega_2})}{\prod_{i=M+1}^{M+N}(1 - a_i e^{-\mathrm{j}\omega_1} - b_i e^{-\mathrm{j}\omega_2})}, \; |a_i| + |b_i| < 1$

Parameters: $\{a_i, b_i\}_{i=1}^{M+N}$

- From log magnitude to sum of exponentials

$$\log |X(e^{\mathrm{j}\boldsymbol{\omega}})| \longrightarrow \sum_{\mathbf{n}} a_i^{n_1} b_i^{n_2}$$

- Reconstuct parameters $\{a_i, b_i\}$
  - Annihilating filter
  - Algebraically-coupled matrix pencil method
- Outperforms existing techniques



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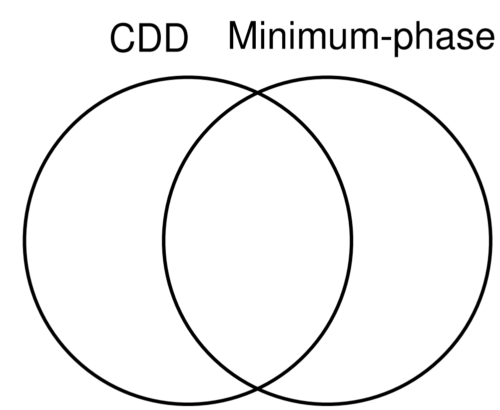
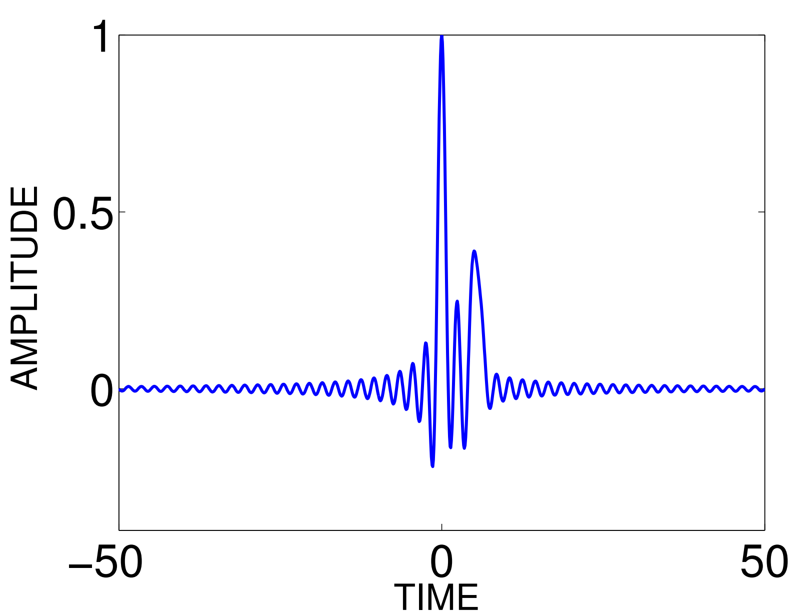
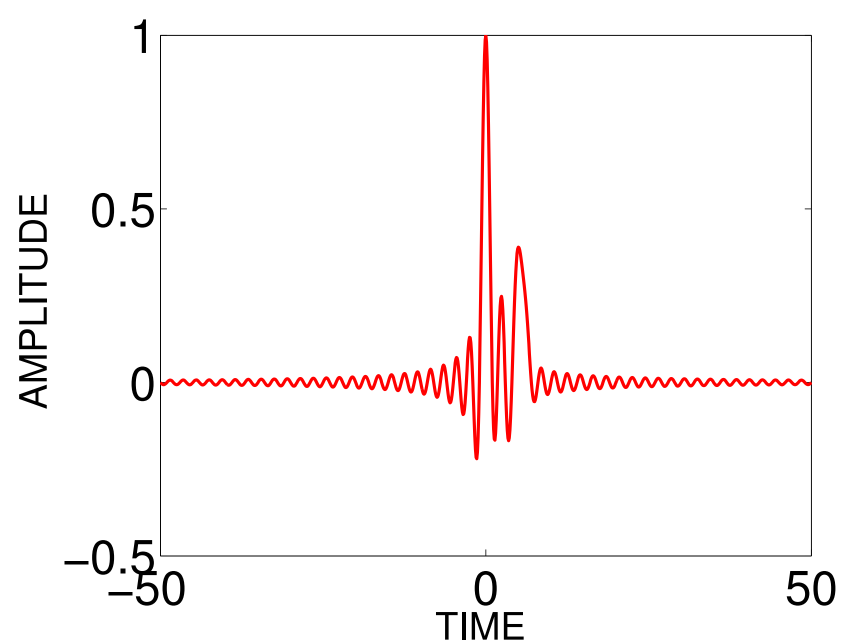
Shift-Invariant Spaces

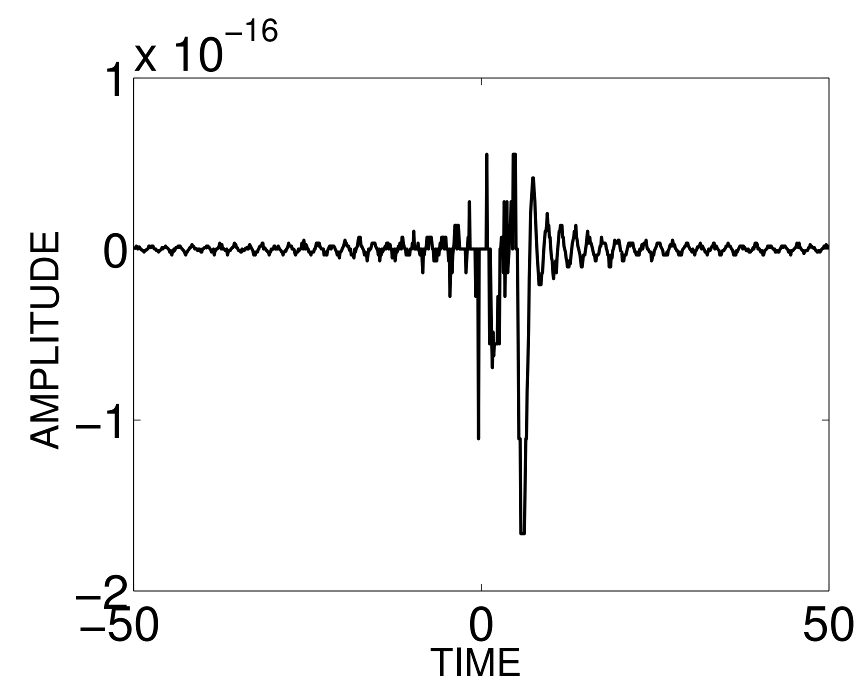
$$f(t) = \sum_{k\in\mathbb{Z}} c_k \phi(t-k)$$

- Problem statement: Given $|\hat{f}(\omega)|$, reconstruct $f(t)$
- $\hat{f}(\omega) = \sum_{k\in\mathbb{Z}} c_k\, \hat{\phi}(\omega) e^{-\mathrm{j}k\omega} = \hat{\phi}(\omega) C(e^{\mathrm{j}\omega})$, if $\{c_k\} \in \ell^1$ and $\hat{\phi} \in L^2$
- Since we know ϕ, the problem is reformulated:
  Given $|C(e^{\mathrm{j}\omega})|$, reconstruct $\{c_k\}_{k\in\mathbb{Z}}$
- CDD: $c_k = \delta_k + a_k$, where $\{a_k\}_{k\in\mathbb{Z}}$ is causal and $\|a_k\|_1 < 1$
- Exact reconstruction of f from $|\hat{f}(\omega)|$ if $\{c_k\}_{k\in\mathbb{Z}}$ is CDD
- There exist CDD signal which are not minimum-phase and vice versa









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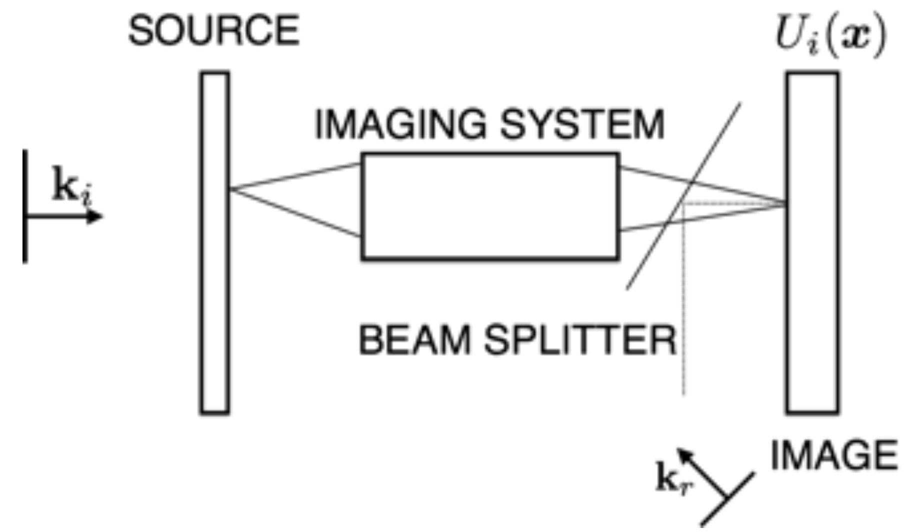
Phase Microscopy

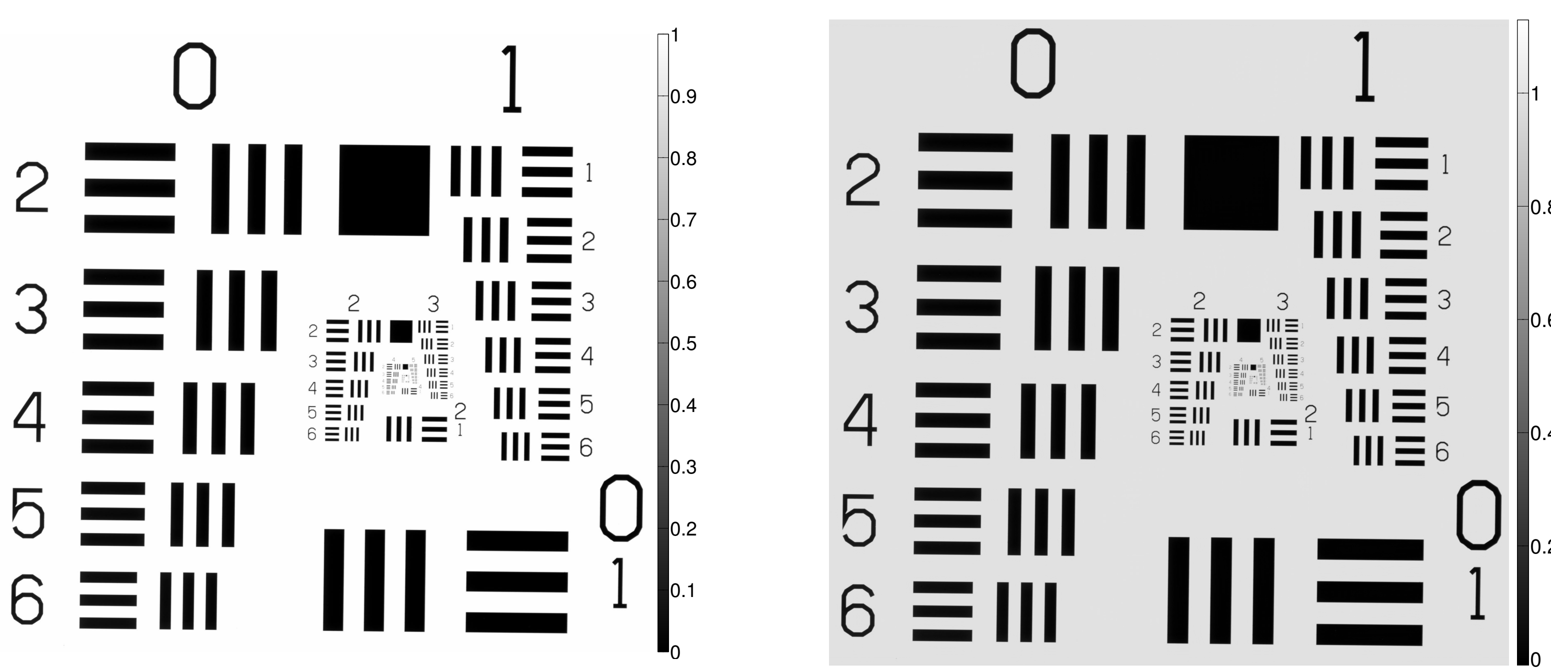
- $U_i(\boldsymbol{x};t) = |U_i(\boldsymbol{x})|\, e^{-\mathrm{j}\,(\langle\omega\rangle t - \langle\mathbf{k}\rangle\cdot\boldsymbol{x} + \phi(\boldsymbol{x},t))}$
- $I(\boldsymbol{x}) = |U_i(\boldsymbol{x}) + U_r(\boldsymbol{x})|^2$
- $\tilde{I}(\boldsymbol{x}) := \frac{I(\boldsymbol{x})}{|U_r(\boldsymbol{x})|^2} = |1 + \tilde{U}_i(\boldsymbol{x})|^2$



Theorem. *Given $I(\boldsymbol{x}) = |U_r(\boldsymbol{x}) + U_i(\boldsymbol{x})|^2$, where $U_r(\boldsymbol{x}) = Re^{\mathrm{j}\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}$ is known, we can exactly reconstruct $U_i(\boldsymbol{x})$, provided that $|U_r(\boldsymbol{x})| > |U_i(\boldsymbol{x})|$ and the spectrum of $U_i(\boldsymbol{x})$ is bandlimited. In particular, if the spectrum of $U_i(\boldsymbol{x})$ is supported within $[-\sigma,\sigma]\times[-\sigma,\sigma]$, and $U_r(\boldsymbol{x}) = Re^{\mathrm{j}\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}$ with $\omega_m > 2\sigma$, $m = 1,2$, we have*

$$\tilde{U}_i(\boldsymbol{x}) = \exp(\mathcal{F}^{-1}\{\mathcal{F}\{\log|1+\tilde{U}_i(\boldsymbol{x})|^2\}\mathbf{1}_{Q_1}\}) - 1,$$

where $\tilde{U}_i(\boldsymbol{x}) = \dfrac{U_i(\boldsymbol{x})}{U_r(\boldsymbol{x})}$, and $\mathbf{1}_{Q_1}$ is the indicator function of the first quadrant.



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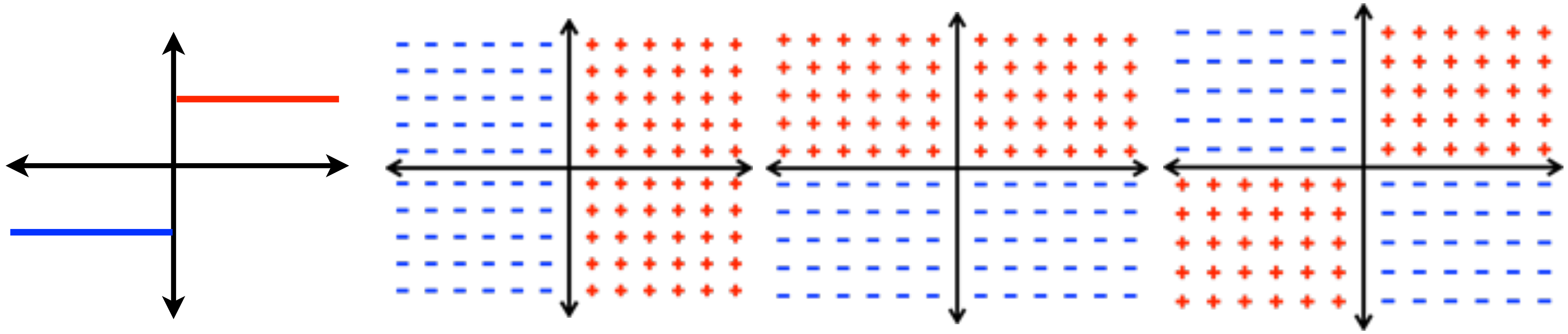
Hilbert Relations

$$\left.\begin{array}{l} x(n) \text{ causal } \Longrightarrow X_r \overset{\mathcal{H}^p}{\longleftrightarrow} X_i \\ f(t) \text{ causal } \Longrightarrow \hat{f}_r \overset{\mathcal{H}^c}{\longleftrightarrow} \hat{f}_i \end{array}\right\} \Longrightarrow \quad \log|X| \overset{\mathcal{H}^p}{\longleftrightarrow} \arg X \text{ for minimum phase}$$

1-D Partial $\mathcal{H}_1$ Partial $\mathcal{H}_2$ Total $\mathcal{H}_{12}$



$f(\boldsymbol{x})$ first-quadrant $\Longrightarrow \hat{f}_r \overset{\mathcal{H}_i^c}{\longleftrightarrow} \hat{f}_i$

Does not carry over to the discrete case. Composite Hilbert transform: $X \circledast K$

$$K(e^{\mathrm{j}\boldsymbol{\omega}}) = \frac{1}{2}\left(\cot\left(\frac{\omega_1}{2}\right)\delta(\omega_2) + \cot\left(\frac{\omega_2}{2}\right)\delta(\omega_1)\right) + \cot\left(\frac{\omega_1}{2}\right) + \cot\left(\frac{\omega_2}{2}\right).$$

$$\left.\begin{array}{l} X(e^{\mathrm{j}\boldsymbol{\omega}}) = \dfrac{\displaystyle\prod_{i=1}^{M}\left(1+Y_i(e^{\mathrm{j}\boldsymbol{\omega}})\right)}{\displaystyle\prod_{i=M+1}^{M+N}\left(1+Y_i(e^{\mathrm{j}\boldsymbol{\omega}})\right)}, \quad |Y_i(e^{\mathrm{j}\boldsymbol{\omega}})| < 1, \forall \boldsymbol{\omega} \\ \hat{f}(\boldsymbol{\omega}) = \dfrac{\displaystyle\prod_{i=1}^{M}\left(1+\hat{g}_i(\boldsymbol{\omega})\right)}{\displaystyle\prod_{i=M+1}^{M+N}\left(1+\hat{g}_i(\boldsymbol{\omega})\right)}, \quad |\hat{g}_i(\boldsymbol{\omega})| \leq \gamma < 1, \forall \boldsymbol{\omega} \end{array}\right\}$$

- Log-magnitude and phase spectra form Hilbert pairs.
- Inverses are stable and allow for exact phase retrieval.
- Generalized minimum-phase signals.

B. A. Shenoy, S. Mulleti, and C. S. Seelamantula, "On Hilbert integral equations, generalized minimum-phase signals, and phase retrieval," Manuscript under revision, IEEE Transactions on Information Theory.

Conclusions

- We generalized the minimum-phase concept to a class of 2-D parametric signals.
- For signals in shift-invariant spaces, we introduced the notion of CDD for the underlying discrete coefficients.
  - We demonstrated exact phase retrieval in this context.
- We showed an application of the phase retrieval technique for CDD signals in quantitative phase microscopy.
- We generalized the Hilbert integral equations to first-quadrant 2-D signals.
  - For 2-D discrete-domain signals, we motivated the need for the composite Hilbert transform, which facilitated the development of 2-D integral equations.
- We introduced the class of generalized minimum-phase signals that satisfy the Hilbert integral equations between their log-magnitude and phase spectra.

Thank You

Any Questions?