# Geometric Tracking Control of Axis-Symmetric, Thrust-Vectored Rigid Bodies on $SE(3)$

*Applied to tail-sitter UAV*


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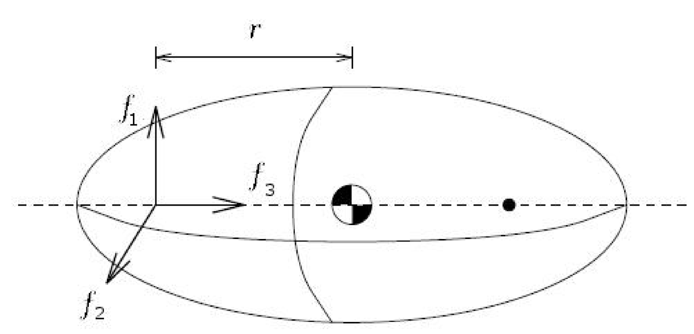
**Abstract**

We propose a trajectory tracking feedback control law for the dynamics of an axis-symmetric rigid body which is subjected to terminal vectored thrust. The control law exploits the geometric structure of the configuration manifold i.e. $SE(3)$, in order to achieve global stability. The control law is intrinsic to the manifold and is thereby free from singularities due to Euclidean parameterizations or input-output decoupling maps. The rigid body dynamics is shown to be differentially flat on the submanifold described by zero axial spin, via the map induced by the Huygen's center of oscillation. Based on the principle of immersion and invariance, a nonlinear proportional derivative feedback law is designed based on the Riemannian structure of $SE(3)/SO(2)$ i.e. the above submanifold. This is augmented with a feedback law for the axial torque which renders the invariant submanifold globally stable, with bounded transients. The overall control law is shown to exponentially stabilize the tracking errors for all initial conditions lying in an open-dense subset of T(SE(3)). The control design is simulated on the dynamics of a tail-sitter UAV and is shown to execute global, aggressive tracking maneuvers.


## Rigid Body Dynamics




$$m\ddot{x} = mge_3 + R\vec{f}$$
$$\dot{R} = R\hat{\Omega}$$
$$J\dot{\Omega} = J\Omega \times \Omega + r[-f_2, f_1, 0]^T + \tau e_3,$$
$$(x, \dot{x}, R, \hat{\Omega}) \in T(SE(3)),$$
$$J = diag(J_1, J_1, J_2)$$

Huygens Center of Oscillation:

$$y := x + \frac{J_1}{rm} Re_3$$

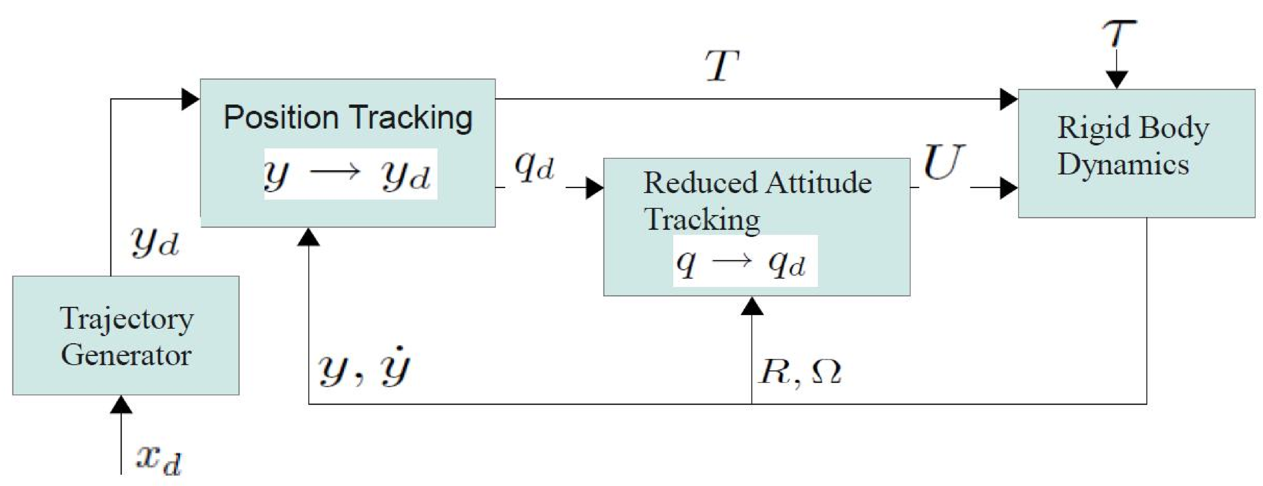
Input Decoupling:

$$m\ddot{y} = mge_3 + TR + \alpha\Omega_3 R[\Omega_1, \Omega_2, 0]$$
$$T = f_3 - \Omega_1^2 - \Omega_2^2$$

## Control Strategy

1. $\Omega_3 \equiv 0 \implies$ Lie-Backlünd Isomorphic to Brunovsky($y$).
2. Design $y_d(t)$ corresponding to $x_d(t)$, bounded internal dynamics.
3. Globally track $y \to y_d$ on $\Omega_3 = 0$ submanifold.
4. Globally stabilize $\Omega_3 = 0$ with bounded transients.

## Geometric Control Design




### Reduced Attitude Tracking

$$\pi : SO(3) \to SO(3)/SO(2)$$
$$q := \pi(R) = Re_3,\ \vec{r}_1 = Re_1,\ \vec{r}_2 = Re_2$$
$$q \in \mathbb{S}^2,\ \vec{r}_1, \vec{r}_2 \in T_q\mathbb{S}^2$$

$$\dot{q} = \Omega_1\vec{r}_2 - \Omega_2\vec{r}_1$$
$$\dot{\Omega}_1 = U_1 + d_1(\Omega)$$
$$\dot{\Omega}_2 = U_2 + d_2(\Omega)$$

$$||d||_2 \leq (||\bar{\Omega}||_2 + ||\bar{\Omega}||_2^2)\Delta$$

### Configuration error on $\mathbb{S}^2$

$$\Psi(q, q_d) = 2 - \frac{2}{\sqrt{2}}\sqrt{1 + q_d^T q}$$
$$\Psi(q, q_d) \in [0, 2],$$
$$\Psi(q, q_d) = 0 \iff q = q_d$$

$$d_1\Psi_{\mathbb{S}^2}(q, qd) = \frac{1}{\sqrt{2}\sqrt{1 + q_d^T q}} q \times (q \times q_d)$$
$$e_q := d_1\Psi_{\mathbb{S}^2}(q, qd)$$
$$||e_q||_2^2 \leq \Psi \leq 2||e_q||_2^2,\ \forall (q, q_d) \in \Psi^{-1}[0, 2)$$

### Transport velocity error on $T\mathbb{S}^2$

$$\mathcal{T}_{\mathbb{S}^2}(q, q_d) : T_{q_d}\mathbb{S}^2 \to T_q\mathbb{S}^2$$
$$\mathcal{T}_{\mathbb{S}^2}(q, q_d).v = (q_d \times v) \times q,\ \forall v \in T_{q_d}\mathbb{S}^2,$$
$$\mathcal{T}_{\mathbb{S}^2}(q, q_d)^*(d_1\Psi_{\mathbb{S}^2}(q, qd)) = -d_2\Psi_{\mathbb{S}^2}(q, qd)$$

$$e_{\dot{q}} := \dot{q} - \mathcal{T}_{\mathbb{S}^2}(q, q_d).\dot{q}_d$$
$$\frac{d}{dt}\Psi(q, q_d) = d_1\Psi_{\mathbb{S}^2}(q, q_d)e_{\dot{q}}$$
$$e_{\dot{q}} \to -k_q g^\sharp(d_1\Psi_{\mathbb{S}^2}),\ k_q > 0$$

### Torque Feedback

$$e_\Omega := ([\Omega_1, \Omega_2]^T - \begin{bmatrix} \langle \vec{r}_2, (\mathcal{T}_{\mathbb{S}^2}(q, q_d).\dot{q}_d - k_q d_1\Psi_{\mathbb{S}^2}) \rangle \\ \langle -\vec{r}_1, (\mathcal{T}_{\mathbb{S}^2}(q, q_d).\dot{q}_d - k_q d_1\Psi_{\mathbb{S}^2}) \rangle \end{bmatrix}$$

$$U_\Delta = \begin{cases} \frac{e_\Omega}{||e_\Omega||_2}, & ||e_\Omega||_2 > tol \\ \frac{e_\Omega}{tol}, & ||e_\Omega||_2 \leq tol \end{cases}$$

$$U(R, \Omega, t) := -\alpha\begin{bmatrix} \langle d_1\Psi_{\mathbb{S}^2}, \vec{r}_2 \rangle \\ \langle d_1\Psi_{\mathbb{S}^2}, -\vec{r}_1 \rangle \end{bmatrix} - k_\Omega e_\Omega + \dot{\Omega}_d - U_\Delta(||\bar{\Omega}||_2 + ||\bar{\Omega}||_2^2)\Delta,$$

$$\tau := (-k_3 - k_2(||U||^2 + \Delta))\Omega_3$$

### Saturated Thrust Feedback

Linear Saturation Function:

$$s\sigma(s) > 0,\ \forall s \neq 0$$
$$\sigma(s) = s,\ \forall |s| \leq a$$
$$|\sigma(s)| \leq b,\ \forall s \in \mathbb{R}$$

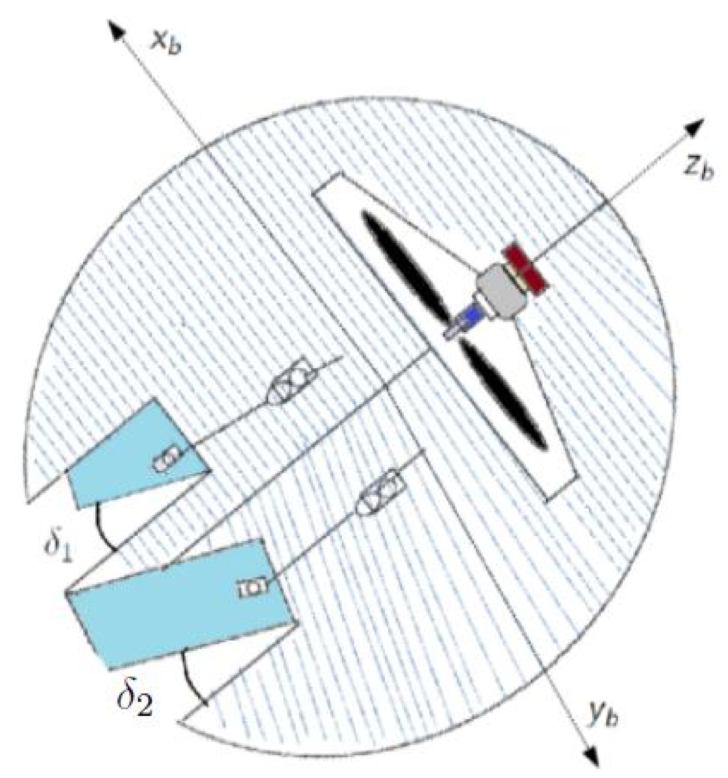
Commanded Vector Thrust:

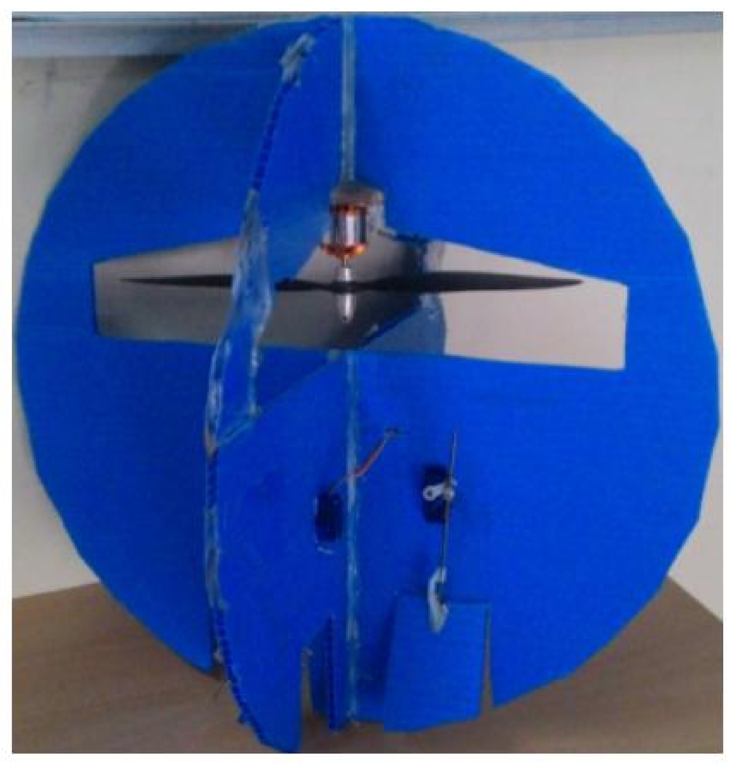
$$\hat{f} = \bar{\sigma}(e_x, e_v) + f_d$$
$$f_d = m\ddot{x}_d + mge_3,$$
$$q_d = \frac{\hat{f}}{||\hat{f}||_2}$$
$$T = H(\langle \hat{f}, Re_3 \rangle)$$

$$\bar{\sigma}(e_x, e_v) = -\begin{bmatrix} \sigma_2\left(\frac{k_1}{k_2}e_{v_1} + \sigma_1\left(k_2 m e_{v_1} + k_1 e_{x_1}\right)\right) \\ \sigma_2\left(\frac{k_1}{k_2}e_{v_2} + \sigma_1\left(k_2 m e_{v_2} + k_1 e_{x_2}\right)\right) \\ \sigma_2\left(\frac{k_1}{k_2}e_{v_3} + \sigma_1\left(k_2 m e_{v_3} + k_1 e_{x_4}\right)\right) \end{bmatrix},$$

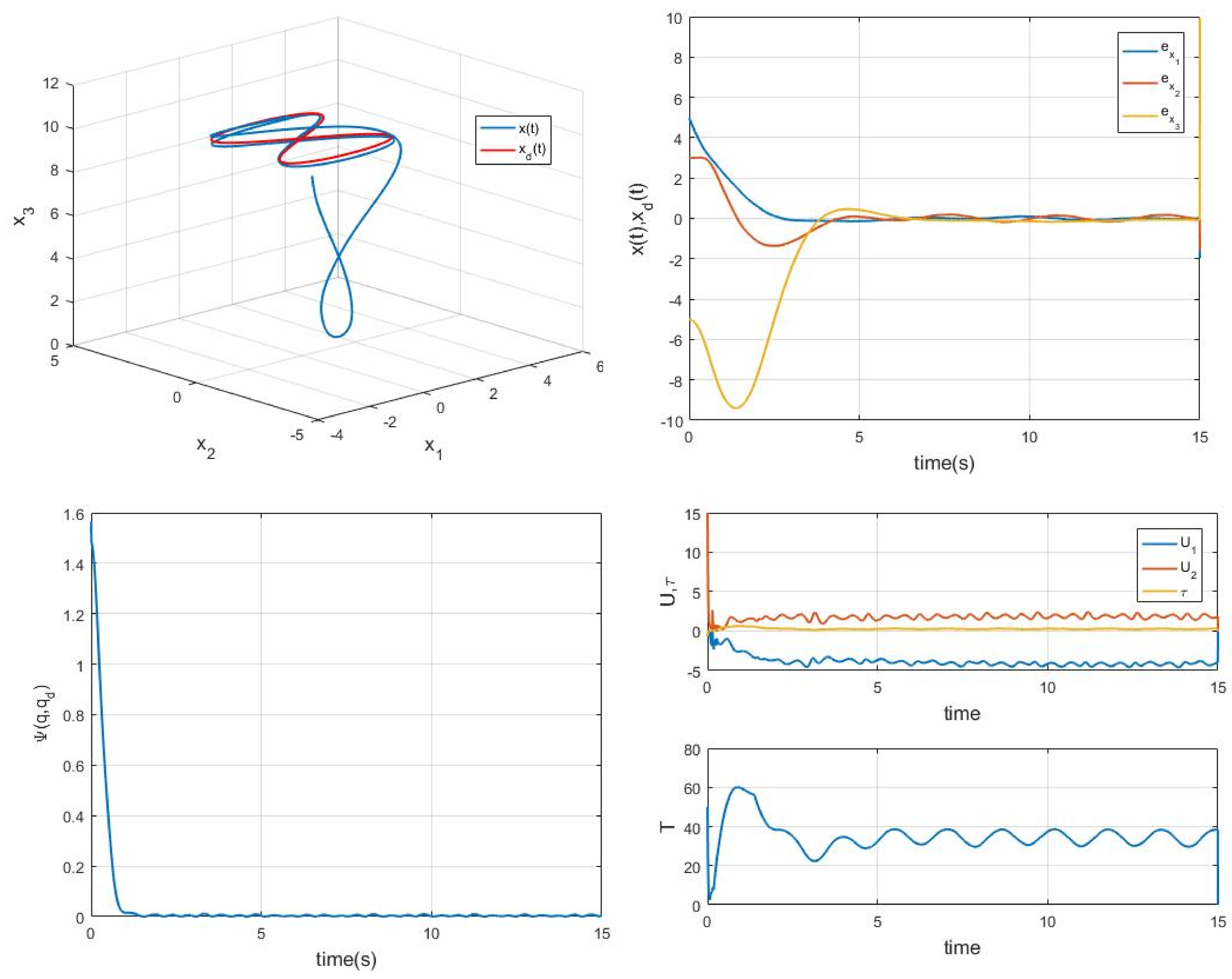
$$b_2 < \inf_{t>0}\{||f_d(t)||_\infty\}$$

## Spherical Tailsitter Drone



### Trajectory tracking simulation




## Conclusions

- Proposed control law free of parameterization singularities
- Global trajectory tracking at Exponential Rate
- Thrust constraints are intrinsically handled
- Robust to bounded uncertainties
- Need to augment with high-bandwidth state-estimators

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## with application to Tail-Sitter Drones

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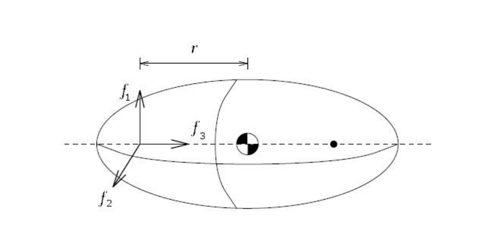
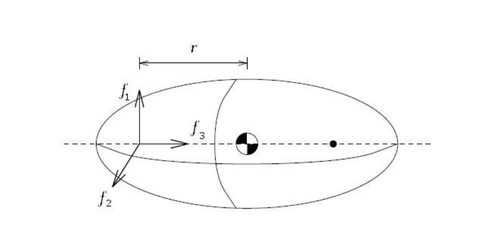
EECS 2017

# Rigid Body Dynamics With $SO(2)$ Symmetry

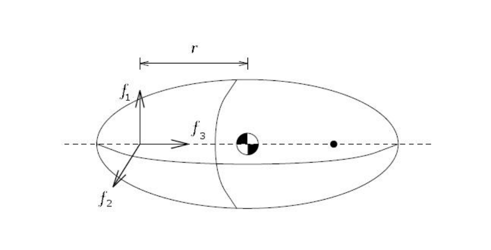


# Rigid Body Dynamics With $SO(2)$ Symmetry




$$
\begin{aligned}
m\ddot{x} &= mge_3 + R\vec{f} \\
\dot{R} &= R\hat{\Omega} \\
J\dot{\Omega} &= J\Omega \times \Omega + r[-f_2, f_1, 0]^T + \tau e_3, \\
(x, \dot{x}, R, \hat{\Omega}) &\in T(SE(3)), \\
J &= diag(J_1, J_1, J_2)
\end{aligned}
\tag{1}
$$

# Rigid Body Dynamics With $SO(2)$ Symmetry

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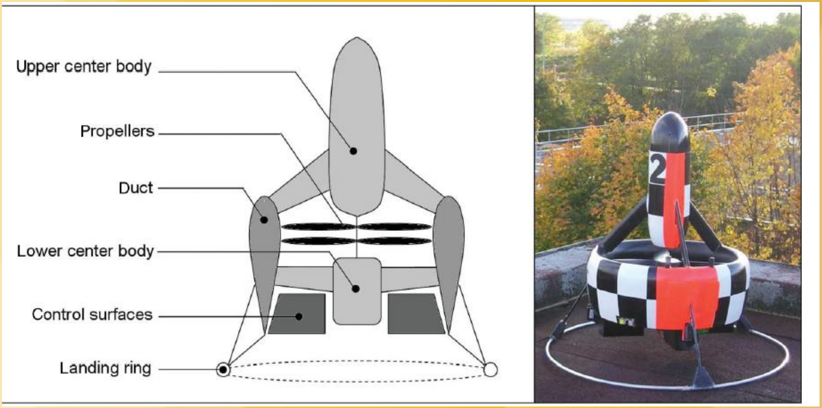
m\ddot{x} &= mge_3 + R\vec{f} \\
\dot{R} &= R\hat{\Omega} \\
J\dot{\Omega} &= J\Omega \times \Omega + r[-f_2, f_1, 0]^T + \tau e_3,
\end{aligned}
\qquad (1)
$$

$$
\begin{aligned}
(x, \dot{x}, R, \hat{\Omega}) &\in T(SE(3)), \\
J &= diag(J_1, J_1, J_2)
\end{aligned}
$$

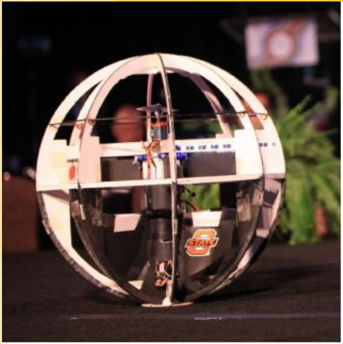
Input-coupling=nonminimum phase system!

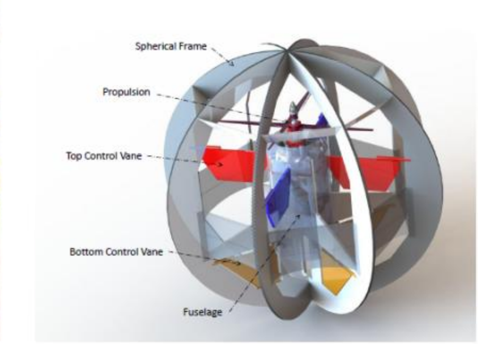
# Hover-Eye Drone




- Propeller induced airflow exerts dynamic pressure on control surfaces
- Control surfaces deflect to generate moments

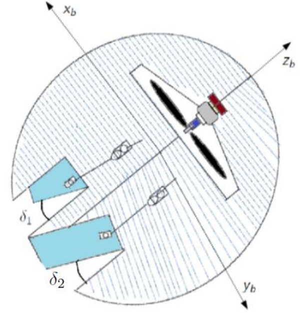
# Japanese Flying Sphere

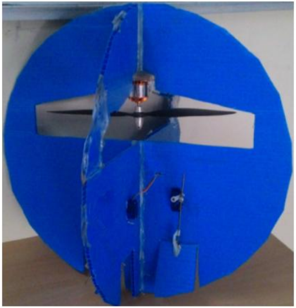





- Two sets of Control Vanes
- Lower CG, Better Stability
- Inefficient aerodynamic design
- Smaller moment arm= Lower agility

# The Sphere-Drone



- Exploits flat-plate aerodynamics in cruise flight
- Capable of rapid transition from hover to cruise mode
- Resistant to collisions, usable in cluttered environment

# Novelty of Proposed Controller

- Intrinsic on $SE(3)$ without Euclidean parameterization
- Global tracking at exponential rate
- Incorporates positive-thrust constraint in mathematical design
- Existing controllers have very conservative region of stability

# Input Decoupling via Differentially Flat outputs

Original Equations:

$$\begin{aligned} m\ddot{x} &= mge_3 + R\vec{f} \\ \dot{R} &= R\hat{\Omega} \\ J\dot{\Omega} &= J\Omega \times \Omega + r[-f_2, f_1, 0]^T + \tau e_3 \end{aligned} \tag{2}$$

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Huygens center of oscillation:

$$y := x + \frac{J_1}{rm} R e_3 \tag{3}$$

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Huygens center of oscillation:

$$y := x + \frac{J_1}{rm} Re_3 \tag{3}$$

Transformed Dynamics:

$$\begin{aligned} m\ddot{y} &= mge_3 + TR + \alpha\Omega_3 R[\Omega_1, \Omega_2, 0] \\ \dot{R} &= R\hat{\Omega} \\ J\dot{\Omega} &= J\Omega \times \Omega + r[-f_2, f_1, 0]^T + \tau e_3, \end{aligned} \tag{4}$$

$$T = f_3 - \Omega_1^2 - \Omega_2^2$$

# Immersion and Invariance Principle

$$
\begin{aligned}
m\ddot{y} &= mge_3 + TR + \alpha\Omega_3 R[\Omega_1, \Omega_2, 0] \\
\dot{R} &= R\hat{\Omega} \\
J\dot{\Omega} &= J\Omega \times \Omega + r[-f_2, f_1, 0]^T + \tau e_3,
\end{aligned}
\tag{5}
$$

- $\Omega_3 \equiv 0 \implies$ Decoupled Dynamics
- Lie-Backlund isomorphic to Brunovsky form in $y$
- Design flat-output trajectory $y_d$ w.r.t $x_d$
- Globally stabilize $y \to y_d$ on Submanifold $\Omega_3 = 0$ (Use $T, f_1, f_2$ )
- Globally stabilize submanifold $\Omega_3 = 0$ (Use $\tau$)

# Reduced Attitude Dynamics on $\Omega_3 = 0$

$$q = Re_3, \vec{r}_1 \quad = \quad Re_1, \vec{r}_2 = Re_2$$

# Reduced Attitude Dynamics on $\Omega_3 = 0$

$$q = Re_3, \vec{r}_1 = Re_1, \vec{r}_2 = Re_2$$

$$\begin{aligned} \dot{q} &= \Omega_1 \vec{r}_2 - \Omega_2 \vec{r}_1 \\ \dot{\Omega}_1 &= U_1 + d_1(\Omega) \\ \dot{\Omega}_2 &= U_2 + d_2(\Omega) \end{aligned}$$

$$q \in \mathbb{S}^2, \; \vec{r}_1, \vec{r}_2 \in T_q\mathbb{S}^2 \tag{6}$$

# Reduced Attitude Dynamics on $\Omega_3 = 0$

$$
\begin{aligned}
q = Re_3, \vec{r}_1 &= Re_1, \vec{r}_2 = Re_2 \\
\\
\dot{q} &= \Omega_1 \vec{r}_2 - \Omega_2 \vec{r}_1 \\
\dot{\Omega}_1 &= U_1 + d_1(\Omega) \\
\dot{\Omega}_2 &= U_2 + d_2(\Omega) \\
\\
q &\in \mathbb{S}^2,\ \vec{r}_1, \vec{r}_2 \in T_q\mathbb{S}^2 \qquad (6)
\end{aligned}
$$

$$
||d||_2 \leq (||\tilde{\Omega}||_2 + ||\tilde{\Omega}||_2^2)\Delta \qquad (7)
$$

# Reduced Attitude Error

Tracking Error via Geodesic Flow on $\mathbb{S}^2$

$$\Psi(q, q_d) = 2 - \frac{2}{\sqrt{2}}\sqrt{1 + q_d^T q} \tag{8}$$

$$\begin{gathered} \Psi(q, q_d) \in [0, 2], \\ \Psi(q, q_d) = 0 \iff q = q_d \end{gathered} \tag{9}$$

# Reduced Attitude Error

Tracking Error via Geodesic Flow on $\mathbb{S}^2$

$$\Psi(q, q_d) = 2 - \frac{2}{\sqrt{2}}\sqrt{1 + q_d^T q} \tag{8}$$

$$\Psi(q, q_d) \in [0, 2], \qquad \Psi(q, q_d) = 0 \iff q = q_d \tag{9}$$

Attitude Error Vector:

$$d_1\Psi_{\mathbb{S}^2}(q, qd) = \frac{1}{\sqrt{2}\sqrt{1 + q_d^T q}} q \times (q \times q_d) \tag{10}$$

$$e_q := d_1\Psi_{\mathbb{S}^2}(q, qd) \tag{11}$$

# Reduced Attitude Error

## Lemma

$$||e_q||_2^2 \leq \Psi \leq 2||e_q||_2^2,\ \forall(q, q_d) \in \Psi^{-1}[0, 2) \tag{12}$$

# Velocity Error via Transport Map

$\mathcal{T}_{\mathbb{S}^2}(q, q_d) : T_{q_d}\mathbb{S}^2 \to T_q\mathbb{S}^2$

$$\mathcal{T}_{\mathbb{S}^2}(q, q_d).v = (q_d \times v) \times q, \ \forall v \in T_{q_d}\mathbb{S}^2 \tag{13}$$

# Velocity Error via Transport Map

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$$\mathcal{T}_{\mathbb{S}^2}(q, q_d).v = (q_d \times v) \times q, \ \forall v \in T_{q_d}\mathbb{S}^2 \tag{13}$$

## Lemma

*The pull-back of the transport map $\mathcal{T}$ satisfies the equation:*

$$\mathcal{T}_{\mathbb{S}^2}(q, q_d)^*(d_1\Psi_{\mathbb{S}^2}(q, qd)) = -d_2\Psi_{\mathbb{S}^2}(q, qd) \tag{14}$$

# Reduced Attitude Error Dynamics

Transported Differential Velocity:

$$e_{\dot{q}} := \dot{q} - \mathcal{T}_{\mathbb{S}^2}(q, q_d).\dot{q}_d \tag{15}$$

$$\begin{aligned} \frac{d}{dt}\Psi(q, q_d) &= d_1\Psi_{\mathbb{S}^2}(q, q_d)\dot{q} + d_2\Psi_{\mathbb{S}^2}(q, q_d)\dot{q}_d \\ &= d_1\Psi_{\mathbb{S}^2}(q, q_d)e_{\dot{q}} \end{aligned} \tag{16}$$

Desired Dynamics:

$$e_{\dot{q}} = -k_q g^{\sharp}(d_1\Psi_{\mathbb{S}^2}), \ \ k_q > 0 \tag{17}$$

# Desired Angular Velocity

$$\Omega_1 \vec{r}_2 - \Omega_2 \vec{r}_1 = \mathcal{T}_{\mathbb{S}^2}(q, q_d).\dot{q}_d - k_q d_1 \Psi_{\mathbb{S}^2} \tag{18}$$

Since *span* $\{\vec{r}_1, \vec{r}_2\} = T_q\mathbb{S}^2$, the above equation admits a unique solution for $\Omega_1$ and $\Omega_2$

$$\Omega_d = \begin{bmatrix} \langle \vec{r}_2, (\mathcal{T}_{\mathbb{S}^2}(q, q_d).\dot{q}_d - k_q d_1 \Psi_{\mathbb{S}^2}) \rangle \\ \langle -\vec{r}_1, (\mathcal{T}_{\mathbb{S}^2}(q, q_d).\dot{q}_d - k_q d_1 \Psi_{\mathbb{S}^2}) \rangle \end{bmatrix} \tag{19}$$

$$e_\Omega \;:=\; ([\Omega_1, \Omega_2]^T - \Omega_d),$$

# Desired Angular Velocity

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$$e_\Omega \;:=\; ([\Omega_1, \Omega_2]^T - \Omega_d),$$

Robustifying Component:

$$U_\Delta = \left\{ \begin{array}{ll} \dfrac{e_\Omega}{||e_\Omega||_2}, & ||e_\Omega||_2 > tol \\ \dfrac{e_\Omega}{tol}, & ||e_\Omega||_2 \leq tol \end{array} \right\} \tag{20}$$

# Reduced attitude tracking control law

## Theorem

*Given a reference trajectory* $q_d(t)$ *which is smooth with bounded derivatives, the control law:*

$$U(R, \Omega, t) := -\alpha \begin{bmatrix} \langle d_1 \Psi_{\mathbb{S}^2}, \vec{r}_2 \rangle \\ \langle d_1 \Psi_{\mathbb{S}^2}, -\vec{r}_1 \rangle \end{bmatrix} - k_\Omega e_\Omega + \dot{\Omega}_d - U_\Delta(||\tilde{\Omega}||_2 + ||\tilde{\Omega}||_2^2)\Delta, \quad (21)$$

*ensures that* $e_q$ *and* $e_\Omega$ *exponentially converge to an arbitrarily small open neighborhood of the origin, for all initial conditions in the open-dense sublevel set* $\Psi^{-1}[0, 2)$ *satisfying:*

$$||e_\Omega(0)||_2 < 2\alpha(2 - \Psi(0)). \quad (22)$$

*Further, the sublevel set* $\Psi^{-1}[0, 2)$ *remains invariant under the flow of (6).*

# Position Tracking with Saturated Thrust

## Definition

Given constants $a$ and $b$ such that $0 < a \leq b$, a function $\sigma : \mathbb{R} \to \mathbb{R}$ is said to be a smooth linear saturation function with limits $(a, b)$, if it is smooth and satisfies:

1. $s\sigma(s) > 0, \ \forall s \neq 0$
2. $\sigma(s) = s, \ \forall |s| \leq a$
3. $|\sigma(s)| \leq b, \ \forall s \in \mathbb{R}$

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1. $s\sigma(s) > 0, \ \forall s \neq 0$
2. $\sigma(s) = s, \ \forall |s| \leq a$
3. $|\sigma(s)| \leq b, \ \forall s \in \mathbb{R}$

Let $\sigma_1$ and $\sigma_2$ be two saturation functions with limits $(a_1, b_1)$ and $(a_2, b_2)$ such that,

$$b_1 < \frac{a_2}{2}. \tag{23}$$

Translational Errors: $e_x := x - x_d, \ e_v = \dot{x} - \dot{x}_d$

# Commanded Thrust Vector

$$\hat{f} = \bar{\sigma}(e_x, e_v) + f_d \tag{24}$$

$$f_d = m\ddot{x}_d + mge_3, \tag{25}$$

$$\bar{\sigma}(e_x, e_v) = -\begin{bmatrix} \sigma_2\left(\frac{k_1}{k_2}e_{v_1} + \sigma_1\left(k_2 m e_{v_1} + k_1 e_{x_1}\right)\right) \\ \sigma_2\left(\frac{k_1}{k_2}e_{v_2} + \sigma_1\left(k_2 m e_{v_2} + k_1 e_{x_2}\right)\right) \\ \sigma_2\left(\frac{k_1}{k_2}e_{v_3} + \sigma_1\left(k_2 m e_{v_3} + k_1 e_{x_4}\right)\right) \end{bmatrix}, \tag{26}$$

and $k_1, \ k_2$ are positive constants.

# Overall stability with bounded attitude Error

### Lemma

*Let $\sigma_1$ and $\sigma_2$ be saturation functions with limits as prescribed in (23). Then, the trajectories of the system*

$$
\begin{aligned}
\dot{y}_1 &= y_2 \\
m\dot{y}_2 &= -\sigma_2((k_1/k_2)y_2 + \sigma_1(k_1 y_1 + k_2 m y_2)) + \xi(t),
\end{aligned}
$$

*enter the linear region of $\sigma_1$ and $\sigma_2$ in a finite time $t_2$ and remain within thereafter if,*
$|\xi(t)| < min\ ((a_2/2) - b_1, a_1),\ \forall t > 0.$

# Feedback Law for Pseudo-Thrust $T$

$$q_d = \frac{\hat{f}}{||\hat{f}||_2} \tag{27}$$

Choose:

$$b_2 < \inf_{t>0} \{||f_d(t)||_\infty\}.$$

Thrust feedback Law:

$$T = H(\langle \hat{f}, Re_3 \rangle). \tag{28}$$

# Exponential Tracking on $SE(3)$

## Theorem

*Consider the control law for $U$ and $T$ as given in (21) and (28) such that the condition (22) is satisfied. Further, define the feedback law for the axial Torque as*

$$\tau := (-k_3 - k_2(||U||^2 + \Delta))\Omega_3 \tag{29}$$

*Define the matrices:*

$$\begin{aligned} W_1 &= \begin{bmatrix} \frac{ck_x}{m}(1-\sin(\theta_0)) & -\frac{ck_v}{2m}(1+\sin(\theta_0)) \\ -\frac{ck_v}{2m}(1+\sin(\theta_0)) & k_v(1-\sin(\theta_0)) - c, \end{bmatrix}, \\ W_2 &= 2\begin{bmatrix} (c/m)||f_d||_2 & 0 \\ a_1 + ||f_d||_2 & 0 \end{bmatrix}, \end{aligned} \tag{30}$$

Given $0 < k_x := k_1$, $0 < k_v := (k_1/k_2) + k_2$, and $\theta_0 < \pi/2$, we choose positive constants $c$, $k_q$, $k_\Omega$, such that

$$c < min\Big\{ \quad k_x k_v (1 - \sin(\theta_0))^2 \left( k_x + \frac{k_v^2(1 + \sin(\theta_0)^2)}{4m} \right)^{-1}, \quad k_v(1 - \sin(\theta_0)), \sqrt{k_x/m} \Big\}, \text{ and}$$

$$min\ (\alpha k_q, k_\Omega) > \frac{4||W_2||^2}{\lambda_{min}(W_1)}. \tag{31}$$

Then, the tracking errors $e_x$, $e_v$, $e_q$, $e_\Omega$, exponentially converge to an arbitrarily small open neighborhood of the origin, for all initial conditions lying in an open-dense subset.

# Trajectory Tracking

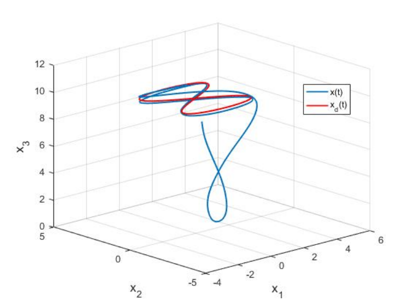


Figure: Trajectory tracking post recovery from inverted pose

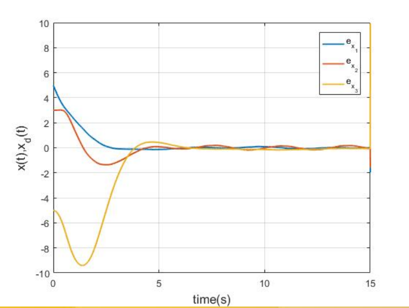


Figure: Exponentially attractive position tracking error

# Trajectory tracking

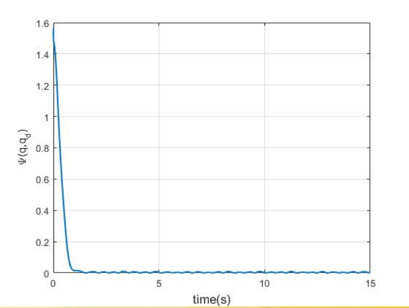


Figure: Reduced attitude error

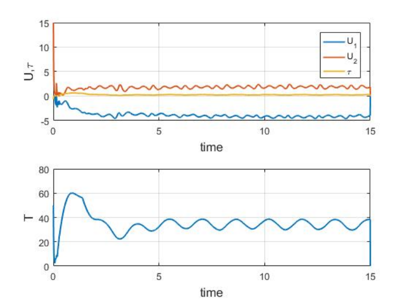


Figure: Thrust and Torque inputs

# Conclusion

- Proposed control law free of parameterization singularities
- Global trajectory tracking at Exponential Rate
- Thrust constraints are intrinsically handled
- Robust to bounded uncertainties
- Need to augment with high-bandwidth state-estimators