# Stabilization schemes for convection dominated scalar problems with different time discretizations in time-dependent domains

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## Objectives

- Convection dominated convection diffusion reaction equation in time-dependent domains
- Small diffusivity induces spurious oscillations in numerical solution
- Stabilization schemes (SUPG, LPS) are considered in time-dependent domains
- ALE approach is used to handle the domain movement
- Numerical analysis with implicit Euler, Crank-Nicolson, backward-difference (BDF2) and higher order discontinuous Galerkin (dG) time discretizations

## Model Problem

### Transient convection diffusion reaction equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \epsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f && \text{in } (0,T]\times\Omega_t,\\ u &= 0 && \text{on } [0,T]\times\partial\Omega_t,\\ u(0,x) &= u_0(x) && \text{in } \Omega_0, \end{aligned}$$

with

$$0 < \mu_0 \le \mu(x) = \left(c - \frac{1}{2}\nabla\cdot\mathbf{b}\right)(t,x), \quad \forall\, x\in\Omega_t$$

**ALE formulation** Let $\hat\Omega$ be a reference domain

$$\mathcal{A}_t : \hat\Omega \to \Omega_t, \qquad \mathcal{A}_t(Y) = x(Y,t), \qquad t\in(0,\mathrm{T})$$

$$\frac{d}{dt}\int_{\Omega_t} u\, dx = \int_{\hat\Omega} \frac{\partial (uJ_{A_t})}{\partial t}\, dY = \int_{\Omega_t}\left(\frac{\partial u}{\partial t} + \mathbf{w}\cdot\nabla u + u\nabla\cdot\mathbf{w}\right) dx$$

### SUPG finite element space discretization

$$\frac{d}{dt}(u,\, v)_t + a_{SUPG}(u_h, v_h)_{h,t} - \int_{\Omega_{h,t}} \nabla\cdot(\mathbf{w}_h u_h)\, v_h\, dx = \int_{\Omega_{h,t}} f v_h\, dx + \sum_{K\in\mathcal{T}_{h,t}} \delta_K \int_K f\,(\mathbf{b}-\mathbf{w}_h)\cdot\nabla v_h\, dK$$

$$a_{SUPG}(u,v)_{h,t} = \epsilon(\nabla u, \nabla v)_{h,t} + (\mathbf{b}\cdot\nabla u, v)_{h,t} + (cu, v)_{h,t} + \sum_{K\in\mathcal{T}_{h,t}} \delta_K(-\epsilon\Delta u + (\mathbf{b}-\mathbf{w}_h)\cdot\nabla u + cu, (\mathbf{b}-\mathbf{w}_h)\cdot\nabla v)_K$$

$$|||u|||_t^2 = \left(\epsilon|u|_{1,t}^2 + \sum_{K\in\mathcal{T}_{h,t}} \delta_K||(\mathbf{b}-\mathbf{w}_h)\cdot\nabla u||_{0,K}^2 + \mu||u||_{0,t}^2\right)$$

$$\delta_K \le \frac{\mu_0}{2||c||_{K,\infty}^2}, \qquad \delta_K \le \frac{h_K^2}{2\epsilon c_{inv}^2},$$

SUPG bilinear form satisfies: $a_{SUPG}(u_h,u_h)_{h,t} \ge \frac{1}{2}|||u_h|||_t^2$.

## Finite difference time stepping

### Implicit Euler time discretization — Unconditionally stable

$$||u_h^{n+1}||_{0,t^{N+1}}^2 + \frac{\Delta t}{2}\sum_{n=0}^{N} |||u_h^{n+1}|||_{t^{n+1/2}}^2 \le ||u_h^0||_{0,t^0}^2 + \frac{2\Delta t}{\mu}\sum_{n=0}^{N} ||f^{n+1/2}||_{0,t^{n+1/2}}^2 + 2\Delta t \sum_{K\in\mathcal{T}_{h,t^{n+1/2}}} \delta_K \sum_{n=0}^{N} ||f^{n+1/2}||_{0,K}^2.$$

### Crank-Nicolson time discretization — Conditionally stable

$$||u_h^{N+1}||_{0,t^{N+1}}^2 + \frac{\Delta t}{4}\sum_{n=0}^{N} |||u_h^{n+1}+u_h^n|||_{t^{n+1/2}}^2 \le \left((1+\Delta t\beta_2^0)||u_h^0||_{0,t^0}^2 + \Delta t\sum_{n=0}^{N}\left(\frac{2}{\mu}+\Delta t\right)||f^{n+1/2}||_{0,t^{n+1/2}}^2\right) \exp\left(\Delta t\sum_{n=1}^{N+1}\frac{\beta_1^n+\beta_2^n}{1-\Delta t(\beta_1^n+\beta_2^n)}\right).$$

### Backward difference time discretization — Conditionally stable

$$||u_h^{n+1}||_{0,t^{n+1}}^2 + ||2u_h^{n+1}-u_h^n||_{0,t^n}^2 + \Delta t\sum_{i=1}^{n+1}|||u_h^i(t)|||_{t^i}^2 \le \left((1+\Delta t\alpha_2^0)||u_h^0||_{0,t^0}^2 + ||2u_h^1-u_h^0||_{0,t^1}^2 + \Delta t\sum_{i=1}^{n+1}\left(\frac{2}{\mu}+\frac{\Delta t}{2}\right)||f^i||_{0,t^i}^2\right) \exp\left(\Delta t\sum_{i=1}^{n+1}\frac{2\alpha_1^i+\alpha_2^i}{1-\Delta t(2\alpha_1^i+\alpha_2^i)}\right).$$

## Discontinuous Galerkin (dG) time stepping

### dG with exact integration in time — Unconditionally stable

$$||U_h(t_N)||_{0,t_N}^2 + \int_0^{t_N} |||U_h(t)|||_{LPS}^2\, dt + \sum_{n=0}^{N-1}||(U_h(t_n^+)) - U_h(t_n))||_{0,t_n}^2 \le ||U_h(0)||_{0,t_0}^2 + \frac{1}{\mu}\int_0^{t_N}||f(t)||_{0,t}^2\, dt.$$

### dG with Radau quadrature in time — Conditionally stable

$$||U_h(t_N)||_{0,t_N}^2 + \sum_{n=0}^{N-1} Q_n^q|||U_h(t)|||_{LPS}^2 + \sum_{n=0}^{N-1}||(U_h(t_n^+)) - U_h(t_n))||_{0,t_n}^2 \le ||U_h(0)||_{0,t_0}^2 + \frac{2}{\mu}\sum_{n=0}^{N-1} Q_n^q||f(t)||_{0,t_n}^2\, dt,$$

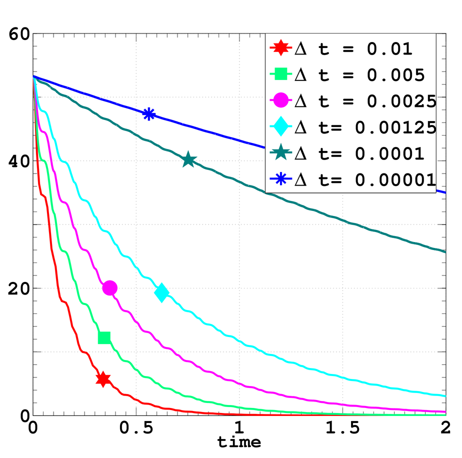
with the condition on time step $\Delta t$ given as,

$$A_n(1+B_{n,2})\Delta t \le \epsilon, \quad \forall\, 0\le n\le N-1.$$

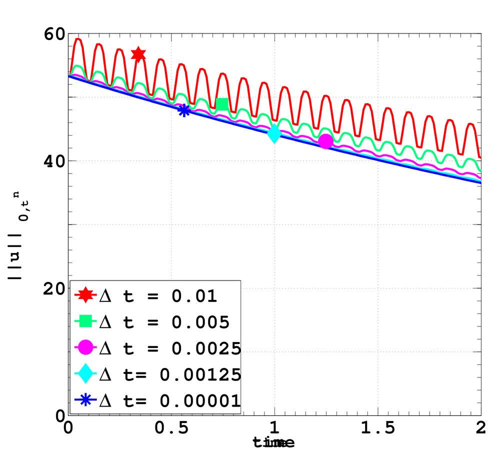
## Numerical Results

**Expanding Square** : $\epsilon = 10^{-2}$, $\mathbf{b} = (0,0)$, $c = 0$

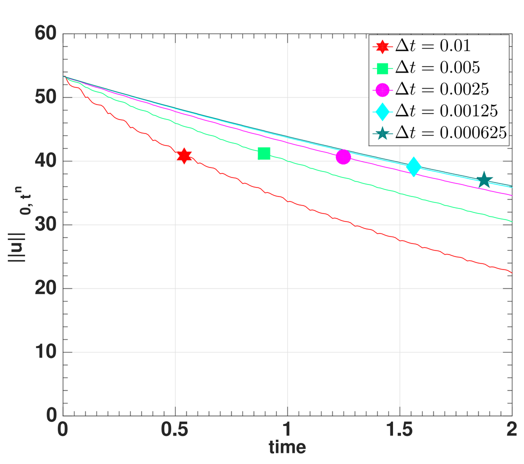
$$x(Y,t) = \mathcal{A}_t(Y) = \{x_1 = Y_1(2-\cos(20\pi t)), \quad x_2 = Y_2(2-\cos(20\pi t))\}$$




IE



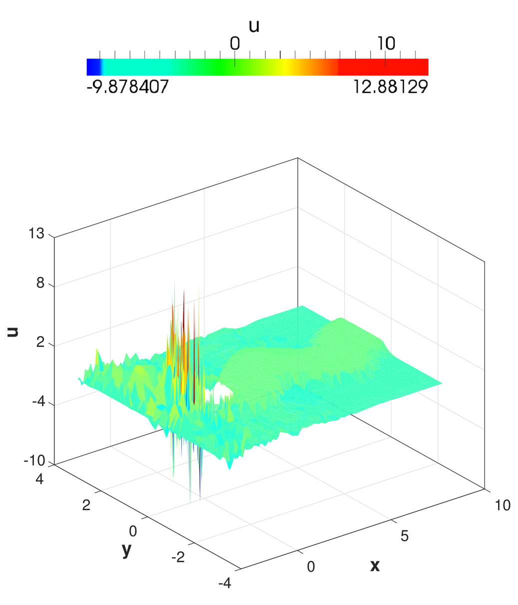

CN



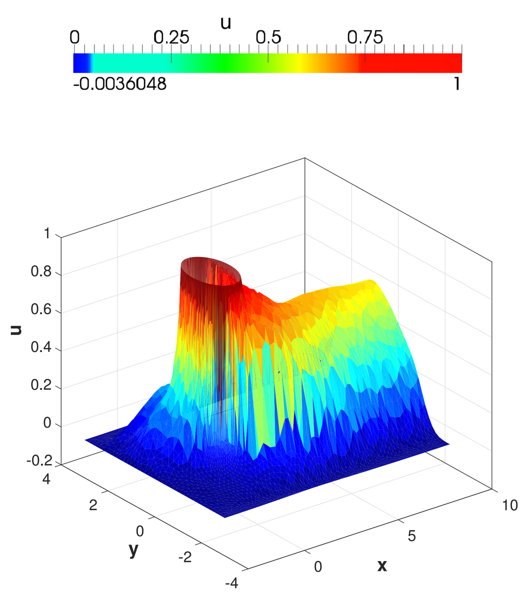

dG(1)

**Hemker example** : $\epsilon = 10^{-8}$, $\mathbf{b} = (1,0)$, $c = 0$

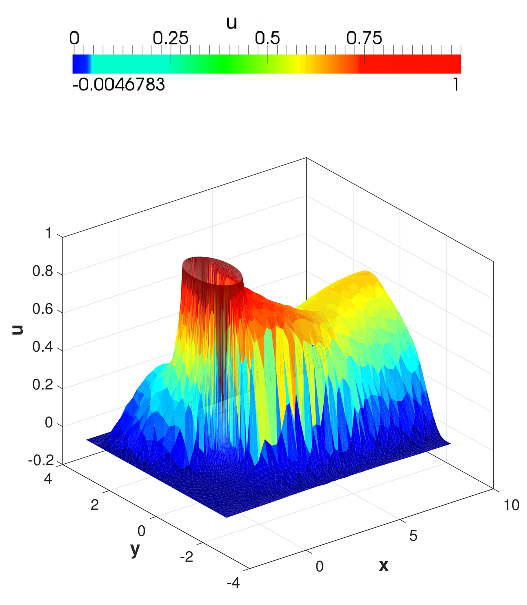
$$x(Y,t) = \mathcal{A}_t(Y) = \{x_1 = Y_1, \quad x_2 = Y_2 + 0.5\sin(2\pi t/5)\}$$



Galerkin



SUPG/IE



SUPG/CN

## References

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[2] Shweta Srivastava and Sashikumaar Ganesan, *On the temporal discretizations of convection dominated convection-diffusion equations in time-dependent domains* , (under review)
[3] Shweta Srivastava and Sashikumaar Ganesan, *Local projection stabilization with discontinuous Galerkin method in time applied to convection dominated problems in time-dependent domains*, submitted

# Stabilization schemes for convection dominated scalar problems with different time discretizations in time-dependent domains

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# OUTLINE

# Convection-diffusion-reaction equation

## Aim

- numerical scheme for convection dominated scalar equations in time-dependent (moving/deforming) domains

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \epsilon \Delta u + \mathbf{b} \cdot \nabla u + cu &= f && \text{in } (0, T] \times \Omega_t, \\
u &= 0 && \text{on } [0, T] \times \partial\Omega_t, \\
u(0, x) &= u_0(x) && \text{in } \Omega_0,
\end{aligned}
$$

with

$$
0 < \mu_0 \leq \mu(x) = \left( c - \frac{1}{2} \nabla \cdot \mathbf{b} \right)(t, x), \quad \forall\, x \in \Omega_t
$$

## Challenges

- solution in time-dependent domain $\Omega_t$
- $0 < \epsilon << ||\mathbf{b}||_\infty$
- contains boundary/interior layers

u - unknown scalar, $t$ - time, $\epsilon$ - diffusion coefficient of u, $\mathbf{b}$ - given convective velocity, $c$ - reaction coefficient, $f$ - source term, $u_0$ - given initial value

# Computations of PDEs with small diffusion

Challenges

- for simplicity consider 1-d case with

$$-\epsilon u'' + bu' = 1 \text{ with } u(0) = u(1) = 0;$$

- solution with $\epsilon = 0, b = 1 \Rightarrow u(x) \notin C[0,1]$
- solution with $\epsilon \geq 0, b = 1$ is

$$u(x) = x - \frac{e^{-\left(\frac{1-x}{\epsilon}\right)}}{1 - e^{-\frac{1}{\epsilon}}} \Rightarrow 0 = \lim_{\epsilon \to 0} \lim_{x \to 1} u(x) \neq \lim_{x \to 1} \lim_{\epsilon \to 0} u(x) = 1$$

- boundary/interior layer problems
- small diffusivity induces spurious oscillations in numerical solution
- stabilization method is considered in moving domains
- time dependent domain makes the analysis even more challenging
- often computational domain is part of the solution

## Numerical scheme

- Arbitrary Lagrangian-Eulerian (ALE) approach for handling time-dependent domains
- stabilization method for spatial discretization of PDEs: SUPG, LPS
- different time discretizations: IE, CN, BDF-2, dG
- First part: ALE-SUPG finite element method for convection-diffusion problems in time-dependent domains: Conservative form (IE, CN time discretization)
- Second part: On the temporal discretizations of convection dominated convection-diffusion equations in time-dependent domains (IE, CN, BDF2 time discretization)
- Third part: Local projection stabilization with discontinuous Galerkin method in time applied to convection dominated problems in time-dependent domains

# ALE APPROACH

ALE mapping

Let $\hat{\Omega}$ be a reference domain, and define a family of bijective ALE mappings

$$\mathcal{A}_t : \hat{\Omega} \to \Omega_t, \qquad \mathcal{A}_t(Y) = x(Y,t), \qquad t \in (0,\mathrm{T})$$

For a function $v \in C^0(\Omega_t)$ on the Eulerian frame, define their corresponding function $\hat{v} \in C^0(\hat{\Omega})$ on the ALE frame by

$$\hat{v} : (0,\mathrm{T}) \times \hat{\Omega} \to \mathbb{R}, \qquad \hat{v} := v \circ \mathcal{A}_t, \qquad \text{with} \qquad \hat{v}(t,Y) = v(t,\mathcal{A}_t(Y))$$

Moreover, the time derivative on the ALE frame is defined as

$$\left.\frac{\partial v}{\partial t}\right|_Y : (0,\mathrm{T}) \times \Omega_t \to \mathbb{R}, \qquad \left.\frac{\partial v}{\partial t}\right|_Y (t,x) = \frac{\partial \hat{v}}{\partial t}(t,Y), \qquad Y = \mathcal{A}_t^{-1}(x)$$

Apply now the chain rule to the time derivative of $v \circ \mathcal{A}_t$ on the ALE frame to get

$$\left.\frac{\partial v}{\partial t}\right|_Y = \frac{\partial v}{\partial t}(t,x) + \left.\frac{\partial x}{\partial t}\right|_Y \cdot \nabla_x v = \frac{\partial v}{\partial t} + \frac{\partial \mathcal{A}_t(Y)}{\partial t} \cdot \nabla_x v = \frac{\partial v}{\partial t} + \mathbf{w} \cdot \nabla_x v$$

where $\mathbf{w}$ is the domain velocity.

# VARIATIONAL FORM

Let $H_0^1(\Omega_t)$ be a subspace of $H^1(\Omega_t)$ in which the functions vanish on the boundary $\partial\Omega_t$. Further, the solution space be

$$V = \left\{ v \in H_0^1(\Omega_t),\ \ v : (0,T] \times \Omega_t \to \mathbb{R},\ \ v = \hat{v} \circ A_t^{-1},\ \ \hat{v} \in H_0^1(\hat{\Omega}) \right\}$$

Non-conservative ALE:

For given $\hat{\Omega}$, $\mathbf{b}$, $\mathbf{w}$, $c$, $u_0$ and $f$, find $u \in V$ such that for all $t \in (0,T]$ and $v \in V$

$$\left( \frac{\partial u}{\partial t},\ v \right)_Y + (\epsilon \nabla u,\ \nabla v)_t + ((\mathbf{b} - \mathbf{w}) \cdot \nabla u,\ v)_t + (cu,\ v)_t = (f,\ v)_t$$

Conservative ALE:

For given $\hat{\Omega}$, $\mathbf{b}$, $\mathbf{w}$, $c$, $u_0$ and $f$, find $u \in V$ such that for all $t \in (0,T]$ and $v \in V$

$$\frac{d}{dt}(u,\ v)_t + (\epsilon \nabla u,\ \nabla v)_t + ((\mathbf{b} - \mathbf{w}) \cdot \nabla u,\ v)_t + ((c - \nabla \cdot \mathbf{w})u,\ v)_t = (f,\ v)_t$$

# Semi-discrete conservative ALE-SUPG Form

For given $\Omega_0$, $u_h(0,x) = u_0(x)$, $\mathbf{b}$, $\mathbf{w}_h$, $c$, and $f$, find $u_h(t,x) \in V_h$ such that for all $t \in (0,T]$ and $v_h \in V_h$,

$$\frac{d}{dt}(u,\ v)_t + a_{SUPG}(u_h, v_h)_{h,t} - \int_{\Omega_{h,t}} \nabla\cdot(\mathbf{w}_h u_h)\ v_h\ dx$$
$$= \int_{\Omega_{h,t}} f v_h\ dx + \sum_{K\in\mathcal{T}_{h,t}} \delta_K \int_K f\ (\mathbf{b}-\mathbf{w}_h)\cdot\nabla v_h\ dK.$$

Inconsistent SUPG is considered, where

$$a_{SUPG}(u,v)_{h,t} = \epsilon(\nabla u, \nabla v)_{h,t} + (\mathbf{b}\cdot\nabla u, v)_{h,t} + (cu, v)_{h,t}$$
$$+ \sum_{K\in\mathcal{T}_{h,t}} \delta_K(-\epsilon\Delta u + (\mathbf{b}-\mathbf{w}_h)\cdot\nabla u + cu, (\mathbf{b}-\mathbf{w}_h)\cdot\nabla v)_K.$$

Here, $\delta_K$ is the SUPG (local) stabilization parameter.

- Inconsistent SUPG as $(\mathbf{b}-\mathbf{w}_h)$ is a function of time.

$$\sum_{K\in\mathcal{T}_{h,\tau}} \int_K \frac{\partial u}{\partial t}\ \delta_K(\mathbf{b}-\mathbf{w}_h)\cdot\nabla v\ dK \neq \sum_{K\in\mathcal{T}_{h,\tau}} \frac{d}{dt}\int_K u\ \ \delta_K(\mathbf{b}-\mathbf{w}_h)\cdot\nabla v\ dK.$$

# Coercivity of bilinear form:

Define mesh dependent norm as

$$|||u|||_t^2 = \left( \epsilon |u|_{1,t}^2 + \sum_{K \in \mathcal{T}_{h,t}} \delta_K ||(\mathbf{b} - \mathbf{w}_h) \cdot \nabla u||_{0,K}^2 + \mu ||u||_{0,t}^2 \right).$$

## Lemma

*Let the discrete form of the assumptions be satisfied. Further, assume that the SUPG parameters satisfy*

$$\delta_K \leq \frac{\mu_0}{2||c||_{K,\infty}^2}, \qquad \delta_K \leq \frac{h_K^2}{2\epsilon c_{inv}^2},$$

*where $c_{inv}$ is a constant used in inverse inequality. Then, the SUPG bilinear form satisfies*

$$a_{SUPG}(u_h, u_h)_{h,t} \geq \frac{1}{2} |||u_h|||_t^2.$$

# Stability of the semi-discrete ALE-SUPG

## Lemma

*The semi-discrete ALE-SUPG solution satisfies*

$$||u_h||_{0,t}^2 + \frac{1}{2}\int_0^T |||u_h|||_t^2 dt \le ||u_h(0)||_{0,t}^2 + \frac{2}{\mu}\int_0^T ||f||_{0,t}^2 \, dt + 2\int_0^T \sum_{K\in\mathcal{T}_{h,t}} \delta_K ||f||_{0,K}^2 \, dt,$$

*which is independent of the mesh velocity* $\mathbf{w}_h$.

Using the Euler expansion, the second term can be written as

$$\int_{\Omega_{h,t}} \frac{\partial u_h}{\partial t}\bigg|_Y u_h \, dx = \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} u_h^2 \, J_{A_t} \, dY - \frac{1}{2}\int_{\hat{\Omega}} u_h^2 \nabla\cdot\mathbf{w}_h \, J_{A_t} \, dY$$

$$= \frac{1}{2}\left(\frac{d}{dt}||u_h||_{0,t}^2 - \int_{\Omega_{h,t}} u_h^2 \nabla\cdot\mathbf{w}_h dx\right)$$

Further, applying the Cauchy-Schwarz and Young's inequalities, the proof is complete.

# Fully discrete scheme

## Temporal discretization

Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a decomposition of the considered time interval $[0, \mathrm{T}]$ into $N$ equal time intervals. Define the discrete ALE mapping for $\tau \in [t^n, t^{n+1}]$ as

$$\mathcal{A}_{h,\Delta t}(Y) = \frac{\tau - t^n}{\Delta t}\mathcal{A}_{h,t^{n+1}}(Y) + \frac{t^{n+1} - \tau}{\Delta t}\mathcal{A}_{h,t^n}(Y),$$

Further, the discrete mesh velocity becomes

$$\hat{\mathbf{w}}_h^{n+1}(Y) = \frac{\mathcal{A}_{h,t^{n+1}}(Y) - \mathcal{A}_{h,t^n}(Y)}{\Delta t}, \qquad \mathbf{w}_h^{n+1} = \hat{\mathbf{w}}_h^{n+1} \circ \mathcal{A}_{h,\Delta t}^{-1}(x)$$

Geometric conservative law (GCL)

$$\int_{\Omega_{t^{n+1}}} \phi_i \, \phi_j \, dx - \int_{\Omega_{t^n}} \phi_i \, \phi_j \, dx = \int_{t^n}^{t^{n+1}} \int_{\Omega_\tau} \phi_i(x) \, \phi_j(x) \, \nabla \cdot \mathbf{w}_h(\tau) \, dx \, d\tau.$$

Since $\mathbf{w}_h$ is piecewise constant in time, GCL becomes

$$\int_{\Omega_{t^{n+1}}} \phi_i \, \phi_j \, dx - \int_{\Omega_{t^n}} \phi_i \, \phi_j \, dx = \Delta t \int_{\Omega_{t^{n+1/2}}} \phi_i(x) \, \phi_j(x) \, \nabla \cdot \mathbf{w}_h \, dx.$$

## Stability estimates for conservative ALE-SUPG with implicit Euler method

fully discrete equation is,

$$\frac{1}{\Delta t}\left[(u_h^{n+1},v_h)_{\Omega_{h,t^{n+1}}} - (u_h^{n},v_h)_{\Omega_{h,t^{n}}}\right] + a_{SUPG}^{n+1/2}(u_h^{n+1},v_h) - \int_{\Omega_{h,t^{n+1/2}}} \nabla\cdot(\mathbf{w}_h u_h^{n+1})\, v_h\, dx$$

$$= \int_{\Omega_{h,t^{n+1/2}}} f^{n+1/2} v_h\, dx + \left(\sum_{K\in\mathcal{T}_{h,t^{n+1/2}}} \delta_K \int_K f^{n+1/2}\,(\mathbf{b}-\mathbf{w}_h)\cdot\nabla v_h\, dK\right).$$

### Lemma

*Assume that $\delta_K \le \frac{\Delta t}{4}$, the discrete ALE-SUPG solution satisfies*

$$||u_h^{n+1}||^2_{0,t^{N+1}} + \frac{\Delta t}{2}\sum_{n=0}^{N} |||u_h^{n+1}|||^2_{t^{n+1/2}}$$

$$\le ||u_h^0||^2_{0,t^0} + \frac{2\Delta t}{\mu}\sum_{n=0}^{N} ||f^{n+1/2}||^2_{0,t^{n+1/2}} + 2\Delta t \sum_{K\in\mathcal{T}_{h,t^{n+1/2}}} \delta_K \sum_{n=0}^{N} ||f^{n+1/2}||^2_{0,K}.$$

*Unconditionally stable: No time step restriction*

## Lemma

*Assume that* $\delta_K \leq \frac{\Delta t}{4}$, *then*

$$\|u_h^{N+1}\|^2_{0,t^{N+1}} + \frac{\Delta t}{4}\sum_{n=0}^{N} |||u_h^{n+1} + u_h^n|||^2_{t^{n+1/2}} \leq \left( (1+\Delta t \beta_2^0)\|u_h^0\|^2_{0,t^0} + \Delta t \sum_{n=0}^{N}\left(\frac{2}{\mu} + \Delta t\right)\|f^{n+1/2}\|^2_{0,t^{n+1/2}}\right) \exp\left(\Delta t \sum_{n=1}^{N+1} \frac{\beta_1^n + \beta_2^n}{1 - \Delta t(\beta_1^n + \beta_2^n)}\right).$$

*Conditionally stable: The estimate is stable with a restriction on* $\Delta t$ *as*

$$\Delta t < \frac{1}{\beta_1^n + \beta_2^n} = \left( ||\nabla \cdot \mathbf{w}_h||_{\infty,t^{n+1/2}} ||J_{\mathcal{A}_{t^{n+1/2},\, t^{n+1}}}||_{\infty,t^{n+1}} + ||\nabla \cdot \mathbf{w}_h||_{\infty,t^{n+1/2}} ||J_{\mathcal{A}_{t^{n},\, t^{n+1/2}}}||_{\infty,t^{n}} \right)^{-1}$$

# Expanding square

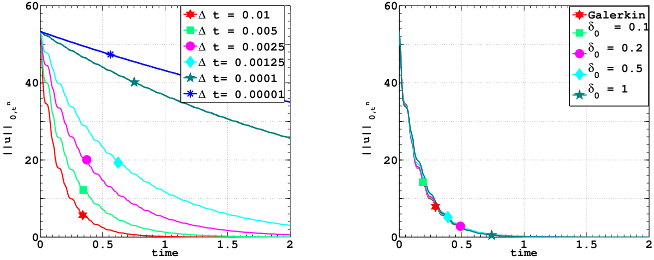
Let $\Omega_0 := (0,1)^2$ be the initial (as well as reference) domain, $\epsilon = 0.01$, $\mathbf{b} = \mathbf{0}$, $c = 0$ and $u_0 = 1600\, Y_1(1 - Y_1)\, Y_2(1 - Y_2)$. Further, the Eulerian coordinate $x(Y,t) \in \Omega_t$ is given by

$$x(Y,t) = \mathcal{A}_t(Y) = \begin{cases} x_1 = Y_1(2 - \cos(20\pi t)) \\ x_2 = Y_2(2 - \cos(20\pi t)), \end{cases} \qquad Y \in \Omega_0$$

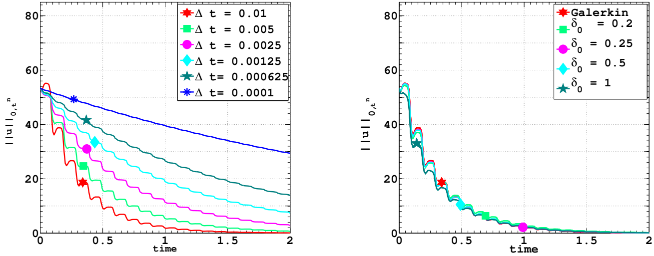
$$\begin{aligned} \frac{\partial u}{\partial t} - 0.01\Delta u - \mathbf{w}\cdot\nabla u &= 0 \\ u &= 0 \\ u(0,x) &= 1600\, x_1(1 - x_1)\, x_2(1 - x_2). \end{aligned}$$

- The domain deformation is given by:

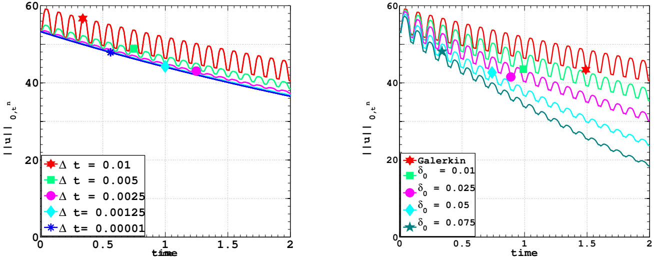
## conservative ALE with IE: Galerkin/SUPG



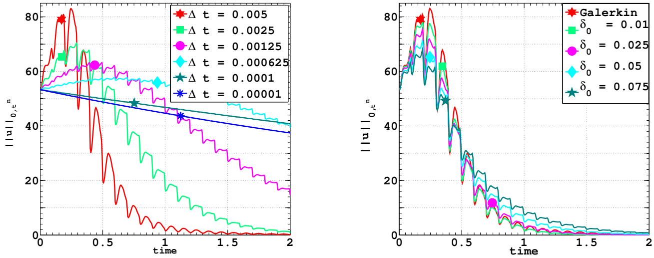

## non-conservative ALE with IE: Galerkin/SUPG



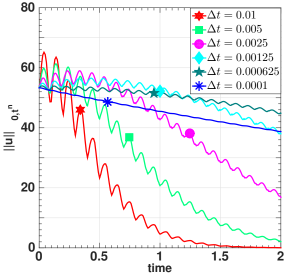
conservative ALE with CN: Galerkin/SUPG

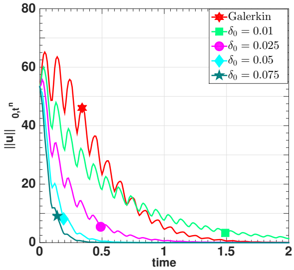



non-conservative ALE with CN: Galerkin/SUPG



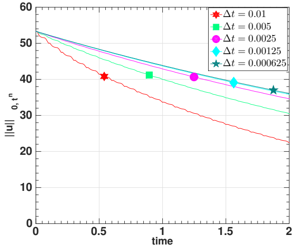
# non-conservative ALE with BDF-2: Galerkin/SUPG

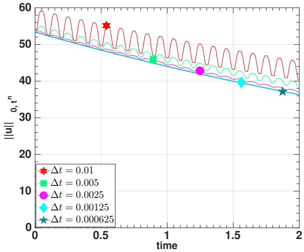



(a) dG(1)

(b) CN




- Second order dG-1 is unconditionally stable, while Crank-Nicolson is conditionally stable only.

# Oscillating disc in a channel: conservative form

Define the computational domain with homogeneous Neumann condition on $\Gamma_N$ as

$$\Omega_t := \{(-3,9)\times(-3,3)\}\setminus\bar{\Omega}_t^S, \quad u_D(x_1,x_2) = \begin{cases} 1 & \text{on } \partial\Omega_t^S, \\ 0 & \text{else} \end{cases}$$

with $\epsilon = 10^{-8}$ and $\mathbf{b}(x_1,x_2) = (1,0)^T$, where

$$\Omega_0^S := \left\{(Y_1,Y_2)\in\mathbb{R}^2;\ Y_1^2+Y_2^2\le 1\right\} \quad \text{and} \quad \Omega_t^S := \{(x_1,x_2)\}\subset\mathbb{R}^2,$$

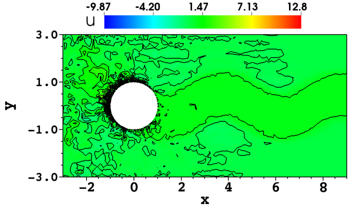
be the reference and time-dependent circular disc with

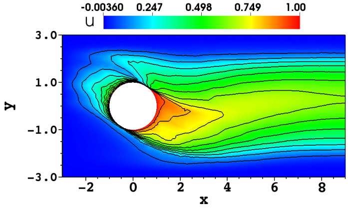
$$x(Y,t) = \mathcal{A}_t(Y) : \begin{cases} x_1 = Y_1 \\ x_2 = Y_2 + 0.5\sin(2\pi t/5). \end{cases}$$

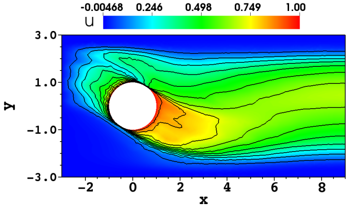
- The domain deformation is given by:

# PERIODICALLY OSCILLATING DISC

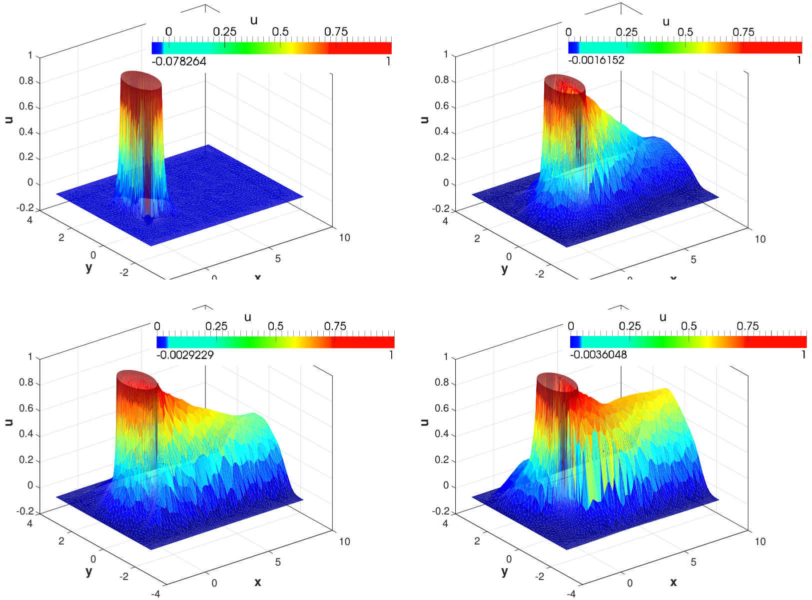
Contour plots of the solution at time $t = 10$: Standard Galerkin with implicit Euler, SUPG with $\delta_0 = 0.1$ and implicit Euler, SUPG with $\delta_0 = 0.1$ and Crank-Nicolson



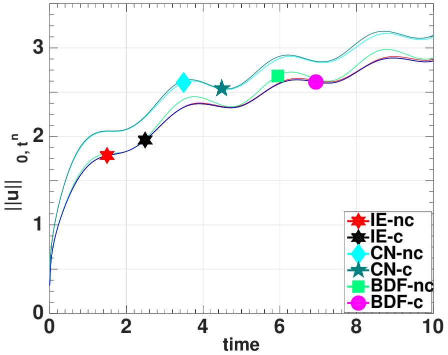



# ALE-SUPG solution with implicit Euler at $t = 0.05, 4, 7, 10$



# OSCILLATING DISC

The variation in the total energy of the system over a period of time with different time discretizations are plotted




The $L^2$-norm of the solution with all the time discretizations for both the conservative and non-conservative case.

## Observations

- semi–discrete in space is unconditionally stable for both conservative and non-conservative case
- conservative ALE-SUPG with IE is unconditionally stable, while all other schemes are conditionally stable with $\Delta t$ depending on ALE map
- non-conservative CN scheme induces high oscillations in the numerical solution compare to other IE and BDF-2 time discretizations
- solutions obtained with the IE and BDF-2 discretizations are more diffusive than the solution of CN discretization
- BDF-2 scheme is more sensitive to the stabilization parameter $\delta_k$ than the other time discretizations
- exact integration in time: stability and error estimates are independent of time step restriction
- Radau quadrature in time: conditionally stable with a time step restriction $\Delta t \leq \frac{\epsilon}{A_n(1+B_{n,2})}$

## References


- S. Ganesan, S. Srivastava: *"ALE-SUPG finite element method for convection-diffusion problems in time-dependent domains: Conservative form "*, Appl. Math. Comput.
- S. Srivastava, S. Ganesan: *"On the temporal discretizations of convection dominated convection-diffusion equations in time-dependent domains "*, (in revision)
- S. Srivastava, S. Ganesan: *"Local projection stabilization with discontinuous Galerkin method in time applied to convection dominated problems in time-dependent domains "*, (submitted)

Thank you for your attention!!!