Name: ______________ GATE Stream: ________ Application Number: __________ Paper: _______

Instructions: Please read carefully. Calculators, mobile phones or other such electronics devices are NOT ALLOWED.

You will have **60 minutes** to answer the questions. This paper has fifteen questions. The first set of five questions are on Programming in C and Data Structures; each of these questions carries 2 points. The remaining questions are on Calculus, Linear Algebra and Matrix Analysis, Probability and Statistics, and Discrete Mathematics and carry one point each.

For multiple choice questions, you must tick all the correct choices to get credit. There is no partial credit for any of the questions. You must write the correct answer in the space provided.

1. (2 points) What are the contents of the array X after the execution of the following code.

```
int i, X[5] = {0,0,0,0,0};

for (i = 0; i < 5; i++)
        X[X[i]]++;
```

Answer:

2. (2 points) Let A be an array containing N integers in increasing order. The following algorithm determines whether there are two distinct numbers in the array whose difference is a specified number S > 0.

```
i = 0; j = 1;
while (j < N) {
   if (?E?) j++;
   else if (A[j]- A[i] == S) break; else i++;
}
if (j<N) printf("yes"); else printf("no");
```

Write the correct expression for ?E?.

Answer:

3. (2 points) What does the following C function return? Assume that a, b, c are distinct integers.

```
int f(int a, int b, int c) {
   if (a>b) if (c<b) return b;
          else return f(a,c,b);
   else return f(b,a,c);
}
```

Answer:

4. (2 points) The following code was written to multiply a 1xN vector by an NxM matrix, where N and M are integer values greater than 0:

```
int i, j;
float vec [N], matrix [N][M], result[M];

for ( i = 0; i < M; i++)                                /* line 1 */
     for (j = 0, result[i] = 0; j < N; j ++)            /* line 2 */
          result [i] += vec[j] * matrix [i] [j];        /* line 3 */
```

Some one comments that this code is wrong but can be corrected by modifying exactly one line. Is this comment correct? If not, justify your answer. If yes, show the modified line.

Answer:

5. (2 points) The following function itoa is supposed to convert an integer into a string

```
void itoa ( int n, char string[] )
{

     static int i;

     if ( n / 10 )
         itoa ( n/10, string );

     else  {
          i = 0;
          if (n < 0)
               string [ i++] = '-';
     }

     string [i++] = abs(n) % 10 + '0';
     string [i] = '\0';
}
```

Some one comments that this function is infinitely recursive for negative values of n. Is this comment correct? If not, justify your answer. If yes, show the corrected function.

Answer:

6. (1 point) Consider the function

$$f(t) = (\mathrm{e}^{-t} - \mathrm{e}^{-2t})u(t-2), \quad \text{where} \quad u(s) := \begin{cases} 1 & \text{when } s \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The maximum value of $f(t)$ is obtained at $t =$______________.

7. (1 point) The function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) := \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where $\mathbb{R}$ is the set of real numbers. Which all of the following hold?

[You must clearly tick all that apply, to get credit.]

○ f is continuous for all $x \in \mathbb{R}$.
○ f is differentiable for all $x \in \mathbb{R}$.
○ f is differentiable for all $x \in \mathbb{R}$, and the derivative is continuous for all $x \in \mathbb{R}$.
○ f is differentiable for all $x \in \mathbb{R}$, and the derivative equals 0 at an infinite number of points.

8. (1 point) Suppose the random vector (X,Y,Z) has a joint distribution with joint density

$$p(x,y,z) = \begin{cases} a & \text{if} \quad 1 \geq x \geq y \geq z \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The value of a is ______________, and the expectation of the random variable X is ______________.

[You must get all the answers correct to get credit.]

9. (1 point) The least-squares solution to the system of equations

$$\begin{aligned} x &= 0 \\ y &= 0 \\ x + y &= 1 \end{aligned}$$

is given by $x =$ ______________ and $y =$ ______________.

[You must get all the answers correct to get credit.]

10. (1 point) Which of the following is/are necessary for a linear system $Ax = b$ to have a solution? (Assume A, x, and b are of appropriate dimensions.)

[You must clearly tick all that apply, to get credit.]

○ b must be linearly independent of the columns of A.
○ A must be invertible.
○ b is in the column space of A.
○ rank($[A \mid b]$) = rank(A).

11. (1 point) A is a 2×2 matrix with two identical eigenvalues 1. Which of the following is/are necessarily TRUE?

    [You must clearly tick all that apply, to get credit.]

    ◯ $A^2 - A$ has two identical eigenvalues 1.
    ◯ A is an identity matrix.
    ◯ $A^2 - A$ is a rank deficient matrix.
    ◯ $A^2 - A$ is a zero matrix.

12. (1 point) Let X and Y be Bernoulli random variables with $P(X=0) = P(Y=0) = 0.7$. The smallest value that $P(X=0, Y=0)$ can take is _____________, and the associated joint distribution is specified by
    $P(X=0, Y=1) =$_____________,
    $P(X=1, Y=0) =$_____________,
    $P(X=1, Y=1) =$_____________.

    [You must get all the answers correct to get credit.]

13. (1 point) A box contains 10 blue balls and 15 green balls. In the first draw, one ball is picked at random and discarded without noticing its colour. The probability that the second draw is green is _____________.

14. (1 point) The variables `x` and `y` are strictly positive real numbers, and `S(x,y)` is a statement that involves the variables `x` and `y`. Consider the following statement:

    ```
    For every x > 0, there exists a y > 0 such that S(x,y) holds.
    ```

    The negation of the above statement is:

    Answer: ____.

15. (1 point) Consider a simple and undirected graph G having 10 vertices, with no edges among vertices numbered 1 through 5, and no edges among vertices numbered 6 through 10. The maximum number of cycles of length five that G can have is _____________.

WRITTEN TEST : M.TECH.(AI) APRIL 2019 : ANSWER KEY TO THE IDENTICAL PAPERS A,B,C,D

1. In each iteration, X[i] = 0 when the loop is entered.
So X[X[i]] = X[0], and this is incremented.
None of the others get incremented
X[0] is incremented 5 times.
Final contents of the array: {5, 0, 0, 0, 0}

2. ?E? = (A[j] − A[i] < S).

3. The middle of the three numbers a, b, c, when arranged in increasing order (or decreasing order).

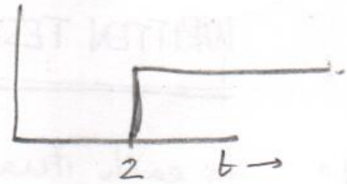
4. Yes, the comment is correct.

```
result[i] += vec[j] * matrix[j][i];
```

To graders: "No, the code is wrong" followed by the correct code line should also be given full points

5. No, the comment is wrong.
$n/10$, $(n/10)/10$, ... will eventually return 0 ≡ FALSE, due to truncation towards 0
The function is not infinitely recursive.

6. Consider the function without $u(t-2)$.



Differentiating, we get

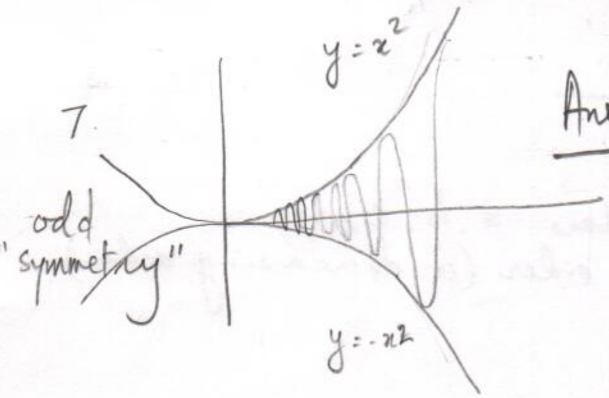
$$-e^{-t} + 2e^{-2t} = -e^{-2t}\left(e^{t} - 2\right).$$

For $t \geq 2$, $e^{t} > 2$, hence derivative is negative for $t > 2$.

Maximum value is at $t = 2$ since $u(t-2) = 1$, since $e^{-2} - e^{-4} > 0$.

<u>Answer</u>: $\boxed{t = 2}$

7.



<u>Answer</u>: $\boxed{A, B, D}$

- f is continuous since at the contentious $x=0$ point, $f(0) = 0$, and $x^2 \sin\frac{1}{x} \longrightarrow 0$ as $x \to 0$.
- Derivative: $2x \sin\left(\frac{1}{x}\right) + x^2 \cos(1/x) \cdot \left(-\frac{1}{x^2}\right) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

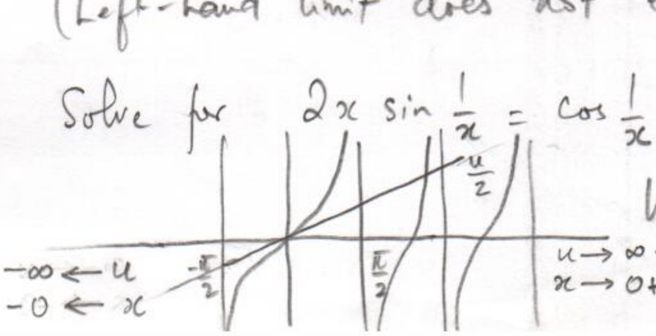
  Differentiable for all $x \neq 0$.

  At $x = 0$: $\displaystyle\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x} = \lim_{x\to 0} x \sin\frac{1}{x} = 0.$

  Differentiable at $x = 0$ as well, so differentiable for all $x \in \mathbb{R}$
- Derivative is not continuous at $x = 0$ since

  $$\lim_{x\to 0} 2x \sin\frac{1}{x} - \cos\frac{1}{x} \neq 0.$$

  (Left-hand limit does not exist)
- Solve for $2x \sin\frac{1}{x} = \cos\frac{1}{x}$, i.e., $\tan\frac{1}{x} = \frac{1}{2}\frac{1}{x}$.



Write $\tan u = \frac{u}{2}$, to get the picture.

<u>Infinite number of solutions</u>

8. $p(x,y,z) = \begin{cases} a & \text{if } 1 \geq x \geq y \geq z \geq 0 \\ 0 & \text{o.w} \end{cases}$

(i) $a \int_0^1 dx \int_0^x dy \int_0^y dz = 1$ to make a probability density.

$$1 = a \int_0^1 dx \int_0^x y = a \int_0^1 dx \, \frac{x^2}{2} = a \cdot \frac{x^3}{3 \cdot 2} \Big|_0^1 = \frac{a}{6}$$

Hence $a = 6$.

(ii) $\mathbb{E}X = a \int_0^1 dx \cdot x \cdot \frac{x^2}{2} = a \, \frac{x^4}{2 \cdot 4} \Big|_0^1 = \frac{6}{8} = \frac{3}{4}$.

Answer: $\boxed{a = 6, \quad \mathbb{E}X = \frac{3}{4}}$

9. Minimise $\underset{x \in \mathbb{R}, y \in \mathbb{R}}{} \; x^2 + y^2 + (x+y-1)^2$

Taking partial derivatives, $\frac{\partial}{\partial x}(\;) = 2x + 2(x+y-1) = 0$

$\frac{\partial}{\partial y}(\;) = 2y + 2(x+y-1) = 0$

This leads to:

$$2x + y = 1$$
$$x + 2y = 1$$

By symmetry, solution satisfies $x = y$. Hence $2x + x = 1$ or $x = \frac{1}{3}$

Answer: $\boxed{x = \frac{1}{3}, \quad y = \frac{1}{3}.}$

10. (i) If b is linearly indep. of cols of A, no linear comb. of columns of A equals b.

$$Ax \neq b \quad \forall x \in \mathbb{R}^n, \text{ if } A = m \times n.$$

(ii) A need not be a square.

(iii) Yes, b must be in the column space of A

(iv) Since b is in column space of A, $[A \,\vdots\, b]$ has the same rank as A.

Answer: $\boxed{C, D}$

11. (i) Take $A = I$. $A^2 - A = 0$; does not have eigenvalues 1.

(ii) Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, not an identity, but have two identical eigenvalues 1

(iii) $A \cdot (A - I)$ has determinant $\det A \times \det(A - I) = 1 \times 0 = 0$.

Yes, rank deficient

(iv) Take $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. $A^2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$.

$$A^2 - A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \neq 0.$$

Answer: $\boxed{C \text{ alone}}$

12. $P[X=0, Y=0] = P[X=0] + P[Y=0] - P[X=0 \text{ or } Y=0]$

$$\geq 0.7 + 0.7 - 1 = 0.4.$$

Try

X \ Y	0	1	
0	0.4	0.3	0.7
1	0.3		
	0.7		

Must be 0.

Answer: $P[X=0, Y=0] = 0.4$

$P[X=0, Y=1] = 0.3$

$P[X=1, Y=0] = 0.3$

$P[X=1, Y=1] = 0$

13. $P[\text{Draw } 2 = \text{Green}] = P[\text{Green}, \text{Green}] + P[\text{Blue}, \text{Green}]$

$$= \frac{15}{25} \times \frac{14}{24} + \frac{10}{25} \times \frac{15}{24}$$

$$= \frac{15}{25}\left[\frac{14}{24} + \frac{10}{24}\right]$$

$$= \frac{15}{25} = \frac{3}{5}.$$

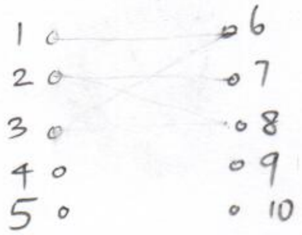
Answer: $\boxed{\frac{3}{5}}$

Note to graders: $\boxed{\frac{15}{25} \text{ is also good. So is } \frac{360}{600}.}$

14. There exists an $x > 0$ such that, for every $y > 0$, $S(x, y)$ does not hold.

Alternative:
$\exists x > 0 : \forall y > 0, \neg S(x, y).$
$\exists x > 0 : \forall y > 0, \ ! S(x, y)$
For some $x > 0$, $S(x, y)$ does not hold for any $y > 0$.

15. No edges among vertices $1, 2, \ldots, 5$
No edges among vertices $6, 7, \ldots, 10$



Bipartite graph.

There are no odd cycles in a bipartite graph

Answer: $\boxed{0}$